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Eigenvectors of the recursion operator and a symmetry structure for the coupled KdV hierarchy

SEN-YUE LOU

Fudan-T. D. Lee Physics Labortary[†]*, Department of Physics, Fudan University, Shanghai, 200433, China*
CCAST (World Laboratory), P. O. Box 8730, Beijing 100080, P. R. China
Department of Physics, Ningbo Normal College, Ningbo 315211, P. R. China[††]
Institute of Theoretical Physics, Academia Sinica, Beijing 100080, P. R. China

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Abstract

It is shown that eigenvectors of the recursion operator L with the eigenvalue λ_i and the inverse of the recursion operator $L_i \equiv L - \lambda_i$ for the coupled KdV hierarchy (CKdVH) can be obtained in terms of squared eigenfunctions of the associated linear problem. The symmetry structure and corresponding infinite dimensional Lie algebras of CKdVH are also given. Using both the local and nonlocal symmetries of CKdVH, one can obtain some exact group invariant solutions and various new infinite-dimensional and finite-dimensional integrable models.

1 Introduction

It is interesting to study the eigenvectors of the recursion operator of nonlinear evolution equations. The authors of ref. [1] pointed out that the discrete eigenvectors of the recursion operator for the Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation can be written in terms of soliton solutions of the isospectral equation, multisolitons can be completely characterized in terms of the discrete spectrum of the recursion operator, and that degeneracy of this spectrum leads to resonance solitons [2, 3].

On the other hand, starting from every symmetry of a $(n+1)$-dimensional integrable model, one can get a new similar model [4]. Furthermore, using the symmetry constraints for a $(n+1)$-dimensional nonlinear evolution equation, one can get also various n-dimensional integrable hierarchies [5–9].

[†]Haida Fellow.
[††]Mailing address.

In this paper, we will study the eigenvectors of the recursion operator and the symmetry structure of the m-component coupled KdV hierarchy (CKdVH). In sec. 2, we point out that derivation of the eigenvectors of the recursion operator L with the eigenvalue λ_i can be casted to find the explicit expression for the inverse of the operator $L_i \equiv L - \lambda_i$. Furthermore, we find that this inverse can be obtained by means of squared eigenfunctions of the associated linear problem of CKdVH using the method given in refs. [4, 10, 11]. In sec. 3, we show that CKdVH possesses four sets of time independent symmetries (which are related to the eigenvectors of the recursion operator with the zero eigenvalue) and one set of time dependent ones. These symmetries constitute an infinite-dimensional Lie algebra which is isomorphic to that of the usual KdV equation. In sec. 4, we discuss the uses of symmetries to get the exact solutions of models. Especially, when the symmetry constraints are used, various finite-dimensional new integrable hierarchies are obtained. The last section is a summary and discussion.

2 Eigenvectors of the recursion operator

For the second-order polynomial eigenvalue problem [12, 13]

$$\phi_{xx} + \sum_{i=0}^{m-1} \lambda^i u_i \phi = \lambda^m \phi \ , \tag{1}$$

in which isospectral flows are shown to possess $(m+1)$ compatible Hamiltonian structures [14], the associated evolution equations can be written as

$$u_{t_n} = L^n u_x \equiv K_n, \tag{2}$$

where $u = (u_0,\ u_1,\ \cdots,\ u_{m-1})^T$ (the superscript T denotes the transposition of a matrix) and L is the recursion operator of the model which has the form

$$L = \begin{pmatrix} 0 & \cdots & 0 & J_0 \\ 1 & \cdots & 0 & J_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & J_{m-1} \end{pmatrix}, \tag{3}$$

$$J_0 = \frac{1}{4}D^2 + u_0 + \frac{1}{2}u_{0x}D^{-1},\ J_i = u_i + \frac{1}{2}u_{ix}D^{-1},\ i = 1,,\ \cdots,\ m-1, \tag{4}$$

with

$$D = \frac{\partial}{\partial x},\ DD^{-1} = D^{-1}D = 1. \tag{5}$$

We call the hierarchy (2) as CKdVH. When $m = 1$ and $m = 2$, (2) becomes the usual KdV hierarchy and the Jaulent–Miodek one, respectively [15, 11].

We now consider the eigenvalue problem of the recursion operator L:

$$(L - \lambda_i)\omega_i = \mathbf{0}\ ,\ i = 1,\ 2,\ \cdots\ ,\ (\omega_i = \omega_{i,0},\ \omega_{i,1},\ \cdots\ ,\ \omega_{i,m-1})^T). \tag{6}$$

Since any constant, say λ_i, corresponds to a trivial strong symmetry of an arbitrary model,

$$(L - \lambda_i) \equiv L_i \tag{7}$$

is also a recursion operator. So to get the eigenvectors of the recursion operator L is equivalent to find the kernel of L_i. The kernel of an operator Ω is a set of vectors σ_n such that $\Omega\sigma_n = 0$. From eq. (6) we know that the eigenvectors of L can be formally written as the inverse of L_i acting on the null vector:

$$\omega_i = L_i^{-1}\mathbf{0} \ , \tag{8}$$

$$L_i L_i^{-1} = L_i^{-1} L_i = \mathbf{1}. \tag{9}$$

Now the problem is transformed to finding the explicit inverse of L_i. For a general operator it is still a quite difficult problem. Fortunately, using the method given in refs. [4, 10, 11], we can easily get the explicit inverse recursion operator for CKdVH.

At first , we set

$$L_i^{-1} = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m-1,0} & A_{m-1,1} & \cdots & A_{m-1,m-1} \end{pmatrix}, \tag{10}$$

such that the operators $A_{a,b}$ $(a,\ b = 0,\ 1,\ 2,\ \cdots\ ,\ m-1)$ are determined by eq. (9). Substituting eq. (10) into eq. (9), we have

$$A_{a-1,b} - \lambda_i A_{a,b} + J_a A_{m-1,b} = \delta_{ab},\ a,\ b\ = 0,\ 1,\ 2,\ \cdots,\ m-1 \tag{11}$$

for the right inverse of L_i and

$$-\lambda_i A_{a,b} + A_{a,b+1} = \delta_{a,b},\ a = 0,\ 1,\ \cdots,\ m-1,\ b = 0,\ 1,\ \cdots,\ m-2\ ,$$

$$\sum_{j=0}^{m-1} A_{a,j} J_j - \lambda_i A_{a,m-1} = \delta_{a,m-1},\ a = 0,\ 1,\ \cdots,\ m-1,\ b = m-1 \tag{12}$$

for the left inverse of L_i, where $A_{-1,b} = 0$, $\delta_{a,a} = 1$ and $\delta_{a,b} = 0$ for $a \neq b$. Solving eq. (11) we get

$$A_{a,b} = \begin{cases} -\displaystyle\sum_{i=0}^{m-2-a} \lambda_i^{i+b} J_{i+1+a} J_{0i}^{-1} + \lambda_i^{m-i-a+b} J_{0i}^{-1}, & a \geq b \\ \displaystyle\sum_{i=0}^{a} \lambda_i^{i+b-1-a} J_i J_{0i}^{-1}, & a < b \end{cases} \tag{13}$$

with

$$J_{0i}^{-1} \equiv \left(\sum_{j=0}^{m-1} \lambda_i^j J_j - \lambda_i^m \right)^{-1} = 4D\phi_i^2 D^{-1}\phi_i^{-2} D^{-1}\phi_i^{-2} D^{-1}\phi_i^2, \tag{14}$$

ϕ_i being the eigenfunction of the isospectral problem (1) for $\lambda = \lambda_i$. Solving eq. (12), one can get the same results as given in eq. (13) with (14). That is to say, both the right

inverse and the left inverse of L_i of CKdVE have the form

$$L_i^{-1} =$$

$$\begin{pmatrix} \lambda_i^{m-1} - \sum_{j=0}^{m-2} \lambda_i^j J_{j+1} & J_0 & \lambda_i J_0 & \cdots & \lambda_i^{m-2} J_0 \\ \lambda_i^{m-2} - \sum_{j=0}^{m-3} \lambda_i^j J_{j+2} & \lambda_i^{m-1} - \sum_{j=0}^{m-2} \lambda_i^{j+1} J_{j+2} & J_0 + \lambda_i J_1 & \cdots & \lambda_i^{m-3}(J_0 + \lambda_i J_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -J_{m-1} + \lambda_i & -\lambda_i J_{m-1} + \lambda_i^2 & -\lambda_i J_{m-1} + \lambda_i^3 & \cdots & \sum_{i=0}^{m-2} \lambda_i^j J_j \\ 1 & \lambda_i & \lambda_i & \cdots & \lambda_i^{m-1} \end{pmatrix} \times$$

$$J_{0i}^{-1} \equiv \Theta_i J_{0i}^{-1} \ . \tag{15}$$

From eq. (12) or (14), we see that the inverse of L_i can be factorized explicitly by means of the isospectral eigenfunction of the original problem for CKdVH.

We now substitute eq. (14) into eq. (8) to get the eigenvectors of L. The result is as follows

$$\begin{aligned} \omega_i = \quad & C_1 \Theta_i \begin{pmatrix} D\phi_i^2 \\ D\phi_i^2 \\ \vdots \\ D\phi_i^2 \end{pmatrix} + C_2 \Theta_i \begin{pmatrix} D\phi_i^2 D^{-1} \phi_i^{-2} D^{-1} \phi_i^{-2} \\ D\phi_i^2 D^{-1} \phi_i^{-2} D^{-1} \phi_i^{-2} \\ \vdots \\ D\phi_i^2 D^{-1} \phi_i^{-2} D^{-1} \phi_i^{-2} \end{pmatrix} + \\ & C_3 \Theta_i \begin{pmatrix} D\phi_i^2 D^{-1} \phi_i^{-2} \\ D\phi_i^2 D^{-1} \phi_i^{-2} \\ \vdots \\ D\phi_i^2 D^{-1} \phi_i^{-2} \end{pmatrix} \equiv C_1 \sigma_{i,0}^{(1)} + C_2 \sigma_{i,0}^{(2)} + C_3 \sigma_{i,0}^{(3)}, \end{aligned} \tag{16}$$

where $C_1,\ C_2$ and C_3 are arbitrary integral functions of time t.

From above discussions we have known that the inverse of L_i and three eigenvectors with the eigenvalue λ_i of L for CKdVH can be expressed by means of the squared eigenfunction of the associated isospectral problem with the same eigenvalue.

3 Symmetries and algebras of CKdVH

On the other hand, the study of symmetries plays an important role in many physical fields. In refs. [4, 10, 11], we showed that for various integrable models like the KdV hierarchy, mKdV hierarchy, Jaulent–Miodek hierarchy and CDGSK hierarchy there exist various sets of infinitely many symmetries. A symmetry of the evolution equation (2), say for $n = 1$, is defined as a solution of its linearized equation:

$$\sigma_t = K_1'\sigma \equiv \frac{\partial}{\partial\varepsilon} K_1(u + \varepsilon\sigma)\mid_{\varepsilon=0} . \tag{17}$$

For the CKdV equation ($n = 1$ in (2)), the linearized operator K_1' has the form

$$K_1' = \begin{pmatrix} A & 0 & 0 & \cdots & 0 & \frac{1}{4}D^3 + B_0 \\ D & A & 0 & \cdots & 0 & B_1 \\ 0 & D & A & \cdots & 0 & B_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & B_{m-2} \\ 0 & 0 & 0 & \cdots & D & A + B_{m-1} \end{pmatrix}, \tag{18}$$

where

$$A = u_{m-1,x} + \frac{1}{2}u_{m-1}D, \qquad B_i = u_i D + \frac{1}{2}u_{ix}. \tag{19}$$

As in the KdV ($m = 1$) [4] and Jaulent–Miodek ($m = 2$) [11] cases, if we restrict the integral functions as constants, especially say zero, one can directly verify that $\sigma_{i0}^{(j)}$ ($j = 1,\ 2,\ 3$) defined in eq. (16) are symmetries of CKdVH (2) by substituting $\sigma_{i0}^{(j)}$ ($j = 1,\ 2,\ 3$) into eq. (17) with (18) due to the hereditary property [16] of the recursion operator. Now together with the generator of the translation group u_x, we have some seed symmetries of CKdVH (2), $u_x,\ \sigma_{i0}^{(j)}$ ($j = 1,\ 2,\ 3$). Now acting with the recursion operator and inverse recursion operators on these seed symmetries, we can get various symmetry hierarchies. However, some symmetries may be expressed by other symmetries via linearly combining them. Actually, it is enough to isolated out the linearly independent symmetries by considering those seeds with zero eigenvalue, because the inverse recursion operator L_i^{-1} with an arbitrary constant λ_i can be expressed by means of L^{-1} with the zero eigenvalue in the following way:

$$L_i^{-1} = (L + \lambda_i)^{-1} = (1 + \lambda_i L^{-1})^{-1} L^{-1} = \sum_{n=0}^{\infty} (-1)^n \lambda_i^n L^{-n-1} \tag{20}$$

and then eq. (16) can be replaced by

$$\omega_i = \sum_{n=0}^{\infty} (-1)^n \lambda_i^n L^{-n} (C_1' \sigma_{0,0}^{(1)} + C_2' \sigma_{0,0}^{(2)} + C_3' \sigma_{0,0}^{(3)}. \tag{21}$$

As in the $m = 1,\ 2$ cases, there exist four sets of linear independent symmetries:

$$\{K_n,\ K_{-n}^{(1)},\ K_{-n}^{(2)},\ K_{-n}^{(3)}\} = \{L^n K_0,\ L^{-n} K_0^{(1)},\ L^{-n} K_0^{(2)},\ L^{-n} K_0^{(3)}\} \tag{22}$$

with

$$K_0 = u_x,\ K_0^{(j)} = \sigma_{i,0}^{(j)}\ |_{\lambda_i=0} = \sigma_{0,0}^{(j)} \qquad (j = 1,\ 2,\ 3), \tag{23}$$

$$L^{-1} = \begin{pmatrix} -J_1J_0^{-1} & 1 & 0 & \cdots & 0 \\ -J_2J_0^{-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -J_{m-1}J_0^{-1} & 0 & 0 & \cdots & 1 \\ J_0^{-1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad J_0^{-1} = 4D\phi_0^2D^{-1}\phi_0^{-2}D^{-1}\phi_0^{-2}D^{-1}\phi_0^2 \tag{24}$$

and $D^{-1}f_x = f$ for an arbitrary f. The isospectral problem (1) for $\lambda = 0$ is reduced to

$$\phi_{0xx} + u_0\phi_0 = 0\ , \tag{25}$$

in which the eigenfunction is dependent on u_0 only.

To look for the full symmetries, we should also find out the so-called τ symmetries (time dependent symmetries) for CKdVH as in the usual KdV case. Using the general method [17], one can easily prove that a set of time dependent symmetries of the CKdVH which is correspondent to the nonisospectral problem of (1) has the form:

$$\tau_n^p = \frac{(p+1)(m+2)}{4}tK_{n+p} + L^nS_0, \quad n = 0,\ \pm1,\ \pm2,\ \cdots\ , \quad p = 0,\ 1,\ 2,\ \cdots\ . \tag{26}$$

$$S_0 = (-\frac{1}{2}u_1,\ -u_2,\ -\frac{3}{2}u_3,\ \cdots,\ -\frac{m-1}{2}u_{m-1},\ \frac{m}{2})^T\ . \tag{27}$$

Starting from these symmetries we get four sets of isospectral integrable hierarchies and one set of a nonisospectral hierarchy. One of the former is just usual CKdVH (2). The other three ones are

$$u_t = K_{-n}^{(1)} = L^{-n}(-J_1,\ -J_2,\ \cdots,\ -J_{m-1},\ 1)^TD\phi_0^2,\ n = 0,\ 1,\ 2,\ \cdots\ , \tag{28}$$

$$u_t = K_{-n}^{(2)} = L^{-n}(-J_1,\ -J_2,\ \cdots,\ -J_{m-1},\ 1)^TD\phi_0^2D^{-1}\phi_0^{-2}D^{-1}\phi_0^{-2},\ n = 0, 1, 2, \cdots, \tag{29}$$

$$u_t = K_{-n}^{(3)} = L^{-n}(-J_1,\ -J_2,\ \cdots,\ -J_{m-1},\ 1)^TD\phi_0^2D^{-1}\phi_0^{-2},\ n = 0,\ 1,\ 2,\ \cdots\ . \tag{30}$$

The nonisospectral hierarchy is given by

$$u_t = \tau_n^p = \frac{(p+1)(m+2)}{4m}tK_{n+p} + L^nS_0,\ n = 0,\ \pm1,\ \pm2,\ \cdots\ ,\ p = 0,\ 1,\ 2,\ \cdots\ . \tag{31}$$

It is worth to point out that

$$L^{-n-1}K_n = L^{n+1}K_{-n}^{(1)} = L^{n+1}K_{-n}^{(2)} = L^{n+1}K_{-n}^{(3)} = 0. \tag{32}$$

That is to say, four sets of time independent symmetries $\mathcal{K} \equiv \{K_n,\ K_{-n}^{(1)},\ K_{-n}^{(2)},\ K_{-n}^{(3)}\}$ are elements of the kernel of the recursion operators $L^n,\ n = \pm1,\ \pm2,\ \cdots$.

Furthermore, we can directly prove that for arbitrary $K \in \mathcal{K}$, the following formula holds

$$(L^{\pm1})'[K] = [K',\ L^{\pm1}] = K'L^{\pm1} - L^{\pm1}K'\ , \tag{33}$$

when $D^{-1}D = 1$ is satisfied (hereafter we always suppose this is true), where

$$(L^{\pm 1})'[K] \equiv \frac{\partial}{\partial \varepsilon} L^{\pm 1}(u + \varepsilon K) \mid_{\varepsilon=0} . \tag{34}$$

That means the operator L is the recursion operator for all four sets of isospectral integrable hierarchies. For the nonisospectral hierarchy, we also have

$$L'[\tau_n] = [\tau_n',\ L] - \frac{1}{2}L^n,\ (L^{-1})'[\tau_n] = [\tau_n',\ L^{-1}] + \frac{1}{2}L^{n-2}. \tag{35}$$

Similar to the $m = 1$ (KdV case) [4], by using eqs. (34) and (35), one can prove that the symmetries $K_n,\ K_{-n}^{(1)},\ K_{-n}^{(2)},\ K_{-n}^{(3)},\ \tau_n$ constitute a Lie algebra

$$[K_k,\ K_n] = [K_k,\ K_{-n}^{(1)}] = [K_k,\ K_{-n}^{(2)}] = [K_k,\ K_{-n}^{(3)}] = 0,$$

$$[K_{-k}^{(1)},\ K_{-n}^{(1)}] = [K_{-k}^{(2)},\ K_{-n}^{(2)}] = [K_{-k}^{(3)},\ K_{-n}^{(3)}] = 0, \tag{36}$$

$$[K_{-k}^{(1)},\ K_{-n}^{(2)}] = -\frac{1}{2}K_{-k-n-1}^{(3)}, \tag{37}$$

$$[K_{-k}^{(1)},\ K_{-n}^{(3)}] = -\frac{1}{2}K_{-k-n-1}^{(1)}, \tag{38}$$

$$[K_{-k}^{(3)},\ K_{-n}^{(2)}] = -\frac{1}{2}K_{-k-n-1}^{(2)}, \tag{39}$$

$$[K_k,\ \tau_n] = \frac{m}{4}(2k+1)K_{k+n-1}\ , \tag{40}$$

$$[K_{-k}^{(1)},\ \tau_n] = -\frac{m}{4}(2k+2-H(n))K_{-k+n-1}^{(1)}\ , \tag{41}$$

$$[K_{-k}^{(2)},\ \tau_n] = -\frac{m}{4}(2k+2+H(n))K_{-k+n-1}^{(2)}\ , \tag{42}$$

$$[K_{-k}^{(3)},\ \tau_n] = -\frac{m}{4}(2k+2+(2n-2)H(n))K_{-k+n-1}^{(3)}\ , \tag{43}$$

$$[\tau_k,\ \tau_n] = \frac{m}{2}(k-n)\tau_{k+n-1}\ , \tag{44}$$

where $H(n) = 0\ (n \leq 0)$, $H(n) = 1\ (n > 0)$ and the Lie product $[F,\ G]$ is defined by

$$[F,\ G] = F'G - G'F \equiv \frac{\partial}{\partial \varepsilon}\{F(u+\varepsilon G) - G(u+\varepsilon F)\} \mid_{\varepsilon=0} . \tag{45}$$

To prove all the conclusions of this section is a quite tedious task. However, the symmetry algebra of CKdVH for arbitrary m is totally isomorphic to that of the usual KdV hierarchy ($m = 1$). The details about the symmetry structure and the symmetry algebras of the KdV hierarchy have been given in ref. [4]. So here we omit concrete proofs for arbitrary m.

4 Group invariant solutions and finite-dimensional integrable models

4.1 Similarity reductions

In principle, starting from an arbitrary symmetry $\sigma(u)$ (infinitesimal transformation) of a model, one can get a corresponding one-parameter group (finite transformation)

$$g_\varepsilon \ : \ u \longrightarrow \bar{u}(u,\ \varepsilon), \tag{46}$$

where $\bar{u} = \bar{u}(u,\ \varepsilon)$ satisfies the initial value problem

$$\begin{cases} \dfrac{d\bar{u}}{d\varepsilon} = \sigma(\bar{u}), \\ \\ \bar{u}\mid_{\varepsilon=0} = u. \end{cases} \tag{47}$$

Using eq. (46) one can get a generalized solution with an arbitrary parameter from the special solution. However, it is quite difficult to get the concrete finite transformation for an arbitrary symmetry though a formal solution of eq. (47) is allowed [18].

Alternatively, for concrete applications, one prefers to use some symmetry constraints to get the so-called group invariant solutions [18, 19] and the lower dimensional integrable models [5–9].

A solution of a model is called group $\mathcal{G}$ invariant, if u is invariant under the action of any $g \in \mathcal{G}$, i.e., $g \circ u = u,\ g \in \mathcal{G}$. Especially, we assume $\mathcal{G} = \{g_\varepsilon \mid \varepsilon \in R\}$ is a one parameter invariant group of the model corresponding to the symmetry $\sigma(u) = \frac{d\bar{u}}{d\varepsilon}\mid_{\varepsilon=0}$. In ref. [18, 19], Tian proved that if g_ε belongs to a one parameter invariant group of an evolution equation, say eq. (2), corresponding to σ, then the solution u is g_ε-invariant if and only if it satisfies $\sigma(u) = 0$. Therefore to look for the group invariant solutions of CKdVH, we only need to solve (2) and

$$\sigma(u) = 0. \tag{48}$$

Eqs. (2) and (48) are compatible [18] and they can be reduced to an ordinary differential equation system. In some special simple cases, it is quite easy to reduce eq. (2) with eq. (48) by using the standard classical Lie Bäclund similarity reduction procedure [20–22].

For convenience, we consider the CKdV equations only for $n = 1$ in eq. (2). The evolution equation (2) for $n = 1$ reads

$$\begin{pmatrix} u_0 \\ \\ u_1 \\ \\ u_2 \\ \vdots \\ u_{m-1} \end{pmatrix}_t = \begin{pmatrix} \frac{1}{4}u_{m-1,xxx} + u_0 u_{m-1,x} + \frac{1}{2}u_{0x}u_{m-1} \\ \\ u_{0x} + u_1 u_{m-1,x} + \frac{1}{2}u_{1x}u_{m-1} \\ \\ u_{1x} + u_2 u_{m-1,x} + \frac{1}{2}u_{2x}u_{m-1} \\ \vdots \\ u_{m-2,x} + u_{m-1}u_{m-1,x} + \frac{1}{2}u_{m-1,x}u_{m-1} \end{pmatrix}. \tag{49}$$

If we take that a symmetry of eq. (49) possesses the form

$$\sigma = aK_0 + cK_1 + \frac{2}{m}b\tau_0 = au_x + cu_t + b(\frac{m+2}{2m}tu_x + \frac{2}{m}S_0) \tag{50}$$

with three arbitrary constants a, b and c, then the corresponding group invariant solutions of eq. (49) are given by eqs. (49) and (48) with (50), i. e.,

$$\begin{pmatrix} cu_{0t} + au_{0x} + \frac{m+2}{2m}btu_{0x} - \frac{1}{m}bu_1 \\ cu_{1t} + au_{1x} + \frac{m+2}{2m}btu_{1x} - \frac{2}{m}bu_2 \\ cu_{2t} + au_{2x} + \frac{m+2}{2m}btu_{2x} - \frac{3}{m}bu_3 \\ \vdots \\ cu_{m-2,t} + au_{m-2,x} + \frac{m+2}{2m}btu_{m-2,x} - \frac{m-1}{m}bu_{m-1} \\ cu_{m-1,t} + au_{m-1,x} + \frac{m+2}{2m}btu_{m-1,x} + b \end{pmatrix} = 0. \tag{51}$$

Finding the generalized solutions of eq. (51) is equivalent to getting the group invariants by solving the characteristic equations

$$\begin{gathered} \frac{du_0}{-\frac{b}{m}u_1} = \frac{du_1}{-\frac{2}{m}bu_2} = \cdots = \frac{du_k}{-\frac{k+1}{m}bu_{k+1}} = \cdots = \\ \frac{du_{m-2}}{-\frac{m-1}{m}bu_{m-1}} = \frac{du_{m-1}}{b} = \frac{dx}{-c - \frac{m+2}{2m}bt} = \frac{dt}{-a} \end{gathered} \tag{52}$$

in the standard classical Lie Bäclund symmetry reduction approach with the symmetry generator being given by eq. (50). The result is

$$\begin{gathered} \xi = ax - \frac{m+2}{4m}bt^2 - ct - x_0, \\ u_{m-1} = U_{m-1}(\xi) - \frac{b}{a}t, \\ u_k = U_k(\xi) + \sum_{i=1}^{m-k-1} \frac{(k+i)!}{i!k!} U_{k+i}(\xi) \left(\frac{bt}{ma}\right)^i - \frac{m!}{k!(m-k)!}\left(\frac{bt}{ma}\right)^{m-k}, \\ k = 0,\ 1,\ \cdots,\ m-2, \end{gathered} \tag{53}$$

where ξ and $U_k(\xi)$, $k = 0,\ 1,\ \cdots,\ m-1$ are group invariants. Finally, the functions $U_k(\xi)$, $k = 0,\ 1,\ \cdots,\ m-1$ should be determined by substituting eq. (53) into the evolution equation (49). After performing some detail calculations, we get the corresponding similarity reduction:

$$\begin{pmatrix} -cU_{0\xi} \\ -cU_{1\xi} \\ -cU_{2\xi} \\ \vdots \\ -cU_{m-2,\xi} \\ -cU_{m-1,\xi} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}a^3U_{m-1,\xi\xi\xi} + aU_0U_{m-1,\xi} + \frac{1}{2}aU_{0\xi}U_{m-1} - \frac{b}{ma}U_1 \\ aU_{0\xi} + aU_1U_{m-1,\xi} + \frac{1}{2}aU_{1\xi}U_{m-1} - \frac{2b}{ma}U_2 \\ aU_{1\xi} + aU_2U_{m-1,\xi} + \frac{1}{2}aU_{2\xi}U_{m-1} - \frac{3b}{ma}U_3 \\ \vdots \\ aU_{m-3,\xi} + aU_{m-2}U_{m-1,\xi} + \frac{1}{2}aU_{m-2,\xi}U_{m-1} - \frac{(m-1)b}{ma}U_{m-2} \\ aU_{m-2,\xi} + aU_{m-1}U_{m-1,\xi} + \frac{1}{2}aU_{m-1,\xi}U_{m-1} + \frac{b}{a} \end{pmatrix}. \tag{54}$$

In particular, for $b = 0$ (54) reduces to the usual travelling wave reduction. To get all the possible similarity reductions, one has to use the direct method [21–23] and/or the nonclassical Lie-Bäklund approach due to Bluman and Cole [24]. However, we do not touch this problem in this paper.

4.2 Finite-dimensional integrable models

In sec. 3, we have found that starting from every symmetry of an integrable model we can get a corresponding (1+1)-dimensional integrable model. Using five sets of symmetries of CKdVH we get five hierarchies of (1+1)-dimensional integrable models. In this subsection we would like to get the corresponding finite-dimensional integrable models using the symmetry constraints.

There exist some ways to restrict infinite-dimensional integrable models to finite-dimensional ones. A natural way is to look for the similarity reductions which has been discussed in the last subsection. Of course, the similarity reduction equation (54) with three arbitrary constants is a finite dimensional integrable model because of integrability of the original evolution equation (49). Actually, using the symmetry constraint

$$\sigma = au_x + cu_t + b\tau_0 = 0, \tag{55}$$

for hierarchies (2) and (28)–(30), where $\tau_0 = C_0 t u_x + S_0$ and constant C_0 is different for every equation of hierarchies (2), (28), (29) and (30), one can get four hierarchies of the corresponding similarity reductions and then four sets of finite-dimensional integrable models can be obtained.

In principle, starting from an arbitrary symmetry constraint

$$\sigma = \sum_{j=1}^{J_0} b_j \prod_{i=1}^{B_j} L_i^{-n_{ij}} \sum_{n=0}^{A_j} a_n K_n + \sum_{j=1}^{J_1} c_j \prod_{i=1}^{C_j} L_i^{-m_{ij}} \sigma_{j,0}^{(1)} + \sum_{j=1}^{J_2} d_j \prod_{i=1}^{D_j} L_i^{-p_{ij}} \sigma_{j,0}^{(2)} +$$
$$\sum_{j=1}^{J_3} f_j \prod_{i=1}^{F_j} L_i^{-q_{ij}} \sigma_{j,0}^{(3)} + \sum_{j=1}^{J_4} g_j \prod_{i=1}^{G_j} L_i^{-r_{ij}} \sum_{n=-R_{1j}}^{R_{2j}} h_n \tau_n = 0, \tag{56}$$

one can get some sets of finite-dimensional integrable hierarchies by searching for the group invariant solutions. In eq. (56), $a_n,\ b_j,\ c_j,\ d_j,\ f_j,\ g_j,$ and h_n are arbitrary constants, $J_0 \sim J_4, A_j,\ B_j,\ C_j,\ D_j,\ F_j,\ G_j,\ R_{1j},\ R_{2j},\ m_{ij},\ n_{ij},\ p_{ij},\ q_{ij}$ and r_{ij} are arbitrary positive integers, $\sigma_{j,0}^{(1)} \sim \sigma_{j,0}^{(3)}$ are seed symmetries given by eq. (16), and all the possible ϕ_i appeared in the inverse recursion operators and seed symmetries are given by eq. (1) with the eigenvalue λ_i. To get the concrete finite-dimensional integrable hierarchies by looking for the similarity reductions is quite difficult except for some special simple cases like $\sigma = K_1,\ K_0,\ \tau_0,\ \tau_1$ and their linear combinations.

For some other special symmetry constraints, one can get the finite-dimensional integrable models in an alternative way by directly combining the symmetry constraint equation and the Lax pair of the model [5–9]. For example, if we take $g_j = 0$ in eq. (56), i.e.,

$$\sum_{j=1}^{J_0} b_j \prod_{i=1}^{B_j} L_i^{-n_{ij}} \sum_{n=0}^{A_j} a_n K_n + \sum_{j=1}^{J_1} c_j \prod_{i=1}^{C_j} L_i^{-m_{ij}} \sigma_{j,0}^{(1)} +$$

$$\sum_{j=1}^{J_2} d_j \prod_{i=1}^{D_j} L_i^{-p_{ij}} \sigma_{j,0}^{(2)} + \sum_{j=1}^{J_3} f_j \prod_{i=1}^{F_j} L_i^{-q_{ij}} \sigma_{j,0}^{(3)} = 0, \tag{57}$$

the system (57) and

$$\phi_{jxx} + \sum_{i=0}^{m-1} \lambda_j^i u_i \phi_j = \lambda_j^m \phi_j \ , \ j = 1, \ 2, \ \cdots \ , \ N,$$

$$N \equiv \max\{J_1 \sim J_3, \ B_1 \sim B_{J_0}, \ C_1 \sim C_{J_1}, \ D_1 \sim D_{J_2}, \ F_1 \sim F_{J_3}\} \tag{58}$$

make a finite dimensional integrable model with dependent variables ϕ_i and u and independent variable x. Two interesting subcases of eqs. (57) and (58) are worthy to be discussed here. (i) If we restrict eq. (57) to have the form

$$K_0 = u_x = \sum_{j=1}^{J_1} c_j \prod_{i=1}^{C_j} L_i^{-m_{ij}} \sigma_{j,0}^{(1)} + \sum_{j=1}^{J_2} d_j \prod_{i=1}^{D_j} L_i^{-p_{ij}} \sigma_{j,0}^{(2)} + \sum_{j=1}^{J_3} f_j \prod_{i=1}^{F_j} L_i^{-q_{ij}} \sigma_{j,0}^{(3)}, \tag{59}$$

then finding u from eq. (59) in terms of ϕ_i and substituting it into eq. (58), we get a finite-dimensional integrable model for the variables ϕ_i only. (ii) More especially, one can use the symmetry constraint for (59) in the form

$$K_0 = u_x = \sum_{j=1}^{J_1} \sigma_{j,0}^{(1)}. \tag{60}$$

In this special case, the authors of the refs. [7] have obtained the Hamiltomian structure of the finite-dimensional integrable model given by (60) and (58) with $N = J_1$. Furthermore, it is interesting that Zeng and Li had pointed out that the symmetry constraint (60) will also reduce the time part of the Lax pairs of CKdVH (2) to the finite-dimensional integrable Hamiltonian systems with time t as the independent variable. Whether there exist (and how to get) the Hamiltonian structures both for the space part and the time part under the more general symmetry constraint (57) is worthy to study in future works.

5 Summary and discussion

In summary, for CKdVH (2) to solve the eigenvalue problem of the recursion operator L is equivalent to solve the original isospectral problem. Concretely, the explicit expression for the inverse of the recursion operator $L_i \equiv L - \lambda_i$ of CKdVH is obtained by means of eigenfunctions of the associated linear problem. Acting with the inverse of the recursion operator L_i on the null vector, the eigenvectors of the recursion operator L with eigenvalue λ_i follow immediately.

Furthermore, if we take all the integral functions as constants (especially say zero) which means that $D^{-1}D = DD^{-1} = 1$ is always used, we find four sets of linearly independent time-independent symmetries and one set of time-dependent symmetries. These symmetries constitute an infinite-dimensional Lie algebra which is isomorphic to that of the usual KdV hierarchy ($m = 1$) [4].

Starting from these sets of symmetries one can get five sets of (1+1)-dimensional integrable models. Furthermore, using the symmetry constraint approach one can get the exact solutions of the original (1+1)-dimensional models from the solutions of finite-dimensional integrable models. One type of similarity reductions (group invariant solutions) is obtained concretely using a simple symmetry constraint. In principle, starting from every symmetry constraint (56) one can get a corresponding group invariant solution though it is very difficult. On the other hand, combining some special symmetry constraint conditions and the x part of the Lax pair of CKdVH, one can get a finite-dimensional integrable model immediately. For a more special symmetry constraint condition (60) studied by Zeng and Li, both the x and t parts of the Lax pair will be reduced to finite-dimensional integrable Hamiltonian systems [7]. The more about these symmetries of the model such as the group invariant solutions under the constraint (56) and whether there exist Hamiltonian structures of the finite-dimensional integrable models reduced from CKdVH (2) under the symmetry constraint (57) or simply (59) should be studied in future works.

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