General Solution of the Universal Equation in n-Dimensional Space

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Abstract

It is shown that general solution of the Universal Equation in n-dimensional space is closely connected with an exactly (but only implicitly) integrable system

$$\frac{\partial \xi_\alpha}{\partial x_n} + \sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} = 0. \tag{0.01}$$

Using the explicit form of the solution of this system it is possible to construct the general solution of the Universal Equation which was found before by the Legendre transformation.

0   Introduction

The original form of the Universal Equation in n-dimensional space is the following one:

$$\det \begin{vmatrix} 0 & \phi_j \\ \phi_i & \phi_{ij} \end{vmatrix} = 0 \quad (1 \leq i, j \leq n) \tag{0.1}$$

where $\phi_{i,j} \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_j}$, $\phi_i \equiv \frac{\partial \phi}{\partial x_i}$, x_i — coordinates of the space [1, 3].

  With respect to each of its independent coordinates (0.1) is an equation of the second order and hence its general solution must depend on two arbitrary functions of $n-1$ independent arguments. Unlike the form of the equation of the Monge-Ampère type considered

in [2], this equation has an additional property (hence the name, Universal) of covariance, i.e. any function of a solution is also a solution.

1   The general construction

If the determinant of an $(n \times n)$-matrix is zero this means that the rows (and columns) of the matrix are linearly dependent. To be precise, a relationship of the following form exists:

$$\sum_j^n \phi_j \alpha_j = 0, \qquad \alpha_0 \phi_i + \sum_j^n \phi_{i,j} \alpha_j = 0, \tag{1.1}$$

where α_j are some functions of coordinates x_i.

The system of equations (1.1) with help of the first scalar equation may be rewritten in the equivalent form

$$\alpha_0 \phi_i = \sum_j \frac{\partial \alpha_j}{\partial x_i} \phi_j. \tag{1.2}$$

Considering ϕ as a function of n independent variables α_i (assuming that $\det_n(\frac{\partial \alpha_j}{\partial x_i}) \neq 0$)) we obtain

$$\frac{\partial \phi}{\partial x_i} = \alpha_0 \frac{\partial \phi}{\partial \alpha_j}. \tag{1.3}$$

From (1.1) and (1.2) we obtain immediately

$$\sum \alpha_j \frac{\partial \phi}{\partial \alpha_j} = 0 \tag{1.4}$$

or that ϕ is a homogeneous function of variables α_i of zero degree. So

$$\phi = R\left(\frac{\alpha_\nu}{\alpha_n}\right) \equiv R(\xi_1, \xi_2, ..., \xi_{n-1}),$$

$$\tag{1.5}$$

$$\frac{\partial R}{\partial x_n} = -\theta \sum \xi_\nu \frac{\partial R}{\partial \xi_\nu}, \quad \frac{\partial R}{\partial x_\mu} = \theta \frac{\partial R}{\partial \xi_\mu}.$$

Here $\xi_\nu = \frac{\alpha_\nu}{\alpha_n}$, $\theta \equiv \frac{\alpha_0}{\alpha_n}$ and from here all Greek indices take values from 1 up to $n-1$.

The condition of compatibility of (1.5) — the equivalence of the second mixed derivatives — will allow us to obtain explicit expressions of $n-1$ new variables ξ_ν as functions of n old coordinates x_i and an explicit expression for the unknown function θ.

The first equation gives

$$\frac{\partial}{\partial x_\alpha} \frac{\partial R}{\partial \xi_\nu} = \frac{\partial}{\partial x_\nu} \frac{\partial R}{\partial \xi_\alpha} \tag{1.6}$$

and the second yields

$$\begin{aligned}
\frac{\partial}{\partial x_\alpha} \frac{\partial \phi}{\partial x_n} &= -\theta \frac{\partial \phi}{\partial x_\alpha} - \sum \xi_\nu \frac{\partial}{\partial x_\alpha} \left(\theta \frac{\partial R}{\partial \xi_\nu}\right) = \\
&\quad -\theta \frac{\partial \phi}{\partial x_\alpha} - \sum \xi_\nu \frac{\partial}{\partial x_\nu} \left(\theta \frac{\partial R}{\partial \xi_\alpha}\right) = \frac{\partial}{\partial x_n} \left(\theta \frac{\partial R}{\partial \xi_\alpha}\right).
\end{aligned} \tag{1.7}$$

From the last equalities we obtain straightforward

$$\sum_\alpha \left(\frac{\partial \xi_\alpha}{\partial x_n} + \sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} \right) \frac{\partial^2 R}{\partial \xi_\alpha \partial \xi_\nu} = - \left[\theta^2 + \frac{\partial \theta}{\partial x_n} + \sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \theta}{\partial x_\beta} \right] \frac{\partial R}{\partial \xi_\nu}. \qquad (1.8)$$

Combining the scalar equation and the system (1.5) and adding to (1.8) we obtain the equation

$$\sum_\alpha \left(\frac{\partial \xi_\alpha}{\partial x_n} + \sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} \right) \frac{\partial R}{\partial \xi_\alpha} = 0. \qquad (1.9)$$

With respect to n variables $\sigma_\alpha = \dfrac{\partial \xi_\alpha}{\partial x_n} + \displaystyle\sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \xi_\alpha}{\partial x_\beta}$ and $\sigma_n = [\theta^2 + \dfrac{\partial \theta}{\partial x_n} + \displaystyle\sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \theta}{\partial x_\beta}]$ and θ (1.8) and (1.9) are a linear system of algebraic equations the matrix of which coincides with the matrix whose determinant set equal to zero is the Universal equation in the space of $(n-1)$-dimensions (compare with (0.1)). Two distinct cases arise in its solution:

$$\det{}_n \begin{pmatrix} 0 & R_\nu \\ R_\mu & R_{\mu,\nu} \end{pmatrix} = 0, \quad \sigma_i \neq 0; \quad \det{}_n \begin{pmatrix} 0 & R_\nu \\ R_\mu & R_{\mu,\nu} \end{pmatrix} \neq 0, \quad \sigma_i = 0. \qquad (1.10)$$

The first possibility is connected with the solution of the Universal equation in the space of $(n-1)$-dimensions. We shall consider the second possibility because the solution of the first one may be obtained from it by reduction.

2 Solution of Equations of a Hydrodynamic Type

The system of equations

$$\frac{\partial \xi_\alpha}{\partial x_n} + \sum_{\beta=1}^{n-1} \xi_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} = 0 \qquad (2.1)$$

from the physical point of view is the equation of velocity flow in free hydrodynamics of $(n-1)$-dimensional space.

In the simplest case $n = 2$ this is the one component equation

$$\frac{\partial \xi}{\partial x_2} + \xi \frac{\partial \xi}{\partial x_1} = 0 \qquad (2.2)$$

which is associated with the name of Monge who first found its general solution in implicit form

$$x_1 - \xi x_2 = f(\xi)$$

where $f(\xi)$ is an arbitrary function of its arguments.

The generalization of this solution to the multidimensional case (2.1) is straightforward [5] and consists in the following construction. Imagine that the system of $n-1$ equations

$$x_\nu - \xi_\nu x_n = Q^\nu(\xi_1, \xi_2, \dots, \xi_n) \qquad (2.3)$$

is solved for the unknown functions ξ_ν. Each solution of this system (2.3) satisfies (2.1).

Indeed, after differentiation of each equation of (2.3) with respect to the variables x_α we obtain

$$\delta_{\alpha,\nu} = \sum \left(\frac{\partial Q^\nu}{\partial x_\beta} + \delta_{\nu,\beta} \right) \frac{\partial \xi_\beta}{\partial x_\alpha}$$

or

$$\frac{\partial \xi_\nu}{\partial x_\alpha} = \left(Ix_n + \frac{\partial Q}{\partial \xi} \right)^{-1}_{\nu,\alpha}. \tag{2.4}$$

The result of differentiation of (2.3) with respect to x_n has as a corollary

$$-\frac{\partial \xi_\nu}{\partial x_n} = \left[\left(Ix_n + \frac{\partial Q}{\partial \xi} \right)^{-1} \xi \right]_\nu. \tag{2.5}$$

The comparison of (2.4) and (2.5) proves that system (2.1) is satisfied. The solution of equation $\theta^2 + \dfrac{\partial \theta}{\partial x_n} + \sum\limits_{\beta=1}^{n-1} \xi_\beta \dfrac{\partial \theta}{\partial x_\beta} = 0$ is obvious and has the form

$$\frac{1}{\theta} = x_n + q(\xi_1, ...\xi_{n-1}), \tag{2.5a}$$

where q is an arbitrary function of $n-1$ arguments ξ.

3   The continuation of the second section

Until now we have used only the condition of compatibility (1.8). Now let us return to equations (1.6). By simple computations using the explicit form for derivatives (2.4) it is easy to show that equations (1.6) may be written in the form

$$\frac{\partial}{\partial \xi_\alpha} \left(\sum \frac{\partial^2 R}{\partial \xi_\beta \xi_\nu} Q^\nu - q\frac{\partial R}{\partial \xi_\alpha} \right) = \frac{\partial}{\partial \xi_\beta} \left(\sum \frac{\partial^2 R}{\partial \xi_\alpha \xi_\nu} Q^\nu - q\frac{\partial R}{\partial \xi_\alpha} \right) \tag{3.1}$$

or

$$\begin{aligned}
\sum \frac{\partial^2 R}{\partial \xi_\beta \xi_\nu} Q^\nu - q\frac{\partial R}{\partial \xi_\alpha} &= \frac{\partial L}{\partial \xi_\beta}, \\
\sum \frac{\partial R}{\partial \xi_\nu} Q^\nu &= L,
\end{aligned} \tag{3.2}$$

where L is an arbitrary function of $n-1$ arguments ξ_ν. If the expression of L from the last equation (3.2) is substituted into $n-1$ first equations keeping in mind (2.4) and (2.5), then the system of equations (1.5) will be produced. In other words (3.2) is exactly equivalent to the system

$$\frac{\partial R}{\partial x_\mu} = \theta \frac{\partial R}{\partial \xi_\mu}.$$

With respect to the functions Q^ν and q (3.2) is a linear system of algebraic equations the matrix of which has determinant not equal to zero (the second case of the (1.9)).

The solution of (3.2) in terms of Cramers determinants has the usual form

$$\begin{pmatrix} q \\ Q \end{pmatrix} = \begin{pmatrix} 0 & R_\xi \\ R_\xi & R_{\xi,\xi} \end{pmatrix}^{-1} \begin{pmatrix} L \\ L_\xi \end{pmatrix} \tag{3.3}$$

and so all of the functions Q^ν, q are represented in terms of only two arbitrary functions L, R and their derivatives up to the first and second orders, respectively.

So with the help of equations (1.5), (2.3) and (3.3) we obtain the solution of the Universal Equation which depends upon two arbitrary functions each of $n-1$ independent arguments and which by the motivation of the Introduction is it's general solution.

4 The simplest examples

$n = 2$

From (1.3) we obtain

$$\frac{\partial \phi}{\partial x_2} = R - \xi_1 \frac{\partial R}{\partial \xi_1}, \qquad \frac{\partial \phi}{\partial x_1} = \frac{\partial R}{\partial \xi_1}. \tag{4.1}$$

The equation (1.9) and its solution are as follows

$$\frac{\partial \xi_1}{\partial x_2} + \xi_1 \frac{\partial \xi_1}{\partial x_1} = 0, \qquad x_1 - \xi_1 x_2 = F(\xi_1). \tag{4.2}$$

In this case the conditions (1.3) are absent and a general solution of the Monge–Ampère equation is determined by the pair of arbitrary functions R, F each of one argument ξ_1.

$n = 3$ — the general case

$$\frac{\partial \phi}{\partial x_3} = R - \xi_1 \frac{\partial R}{\partial \xi_1} - \xi_2 \frac{\partial R}{\partial \xi_2}, \qquad \frac{\partial \phi}{\partial x_1} = \frac{\partial R}{\partial \xi_1}, \qquad \frac{\partial \phi}{\partial x_2} = \frac{\partial R}{\partial \xi_2}. \tag{4.3}$$

The connection between the arguments ξ and x keeping in mind (3.1) takes the form

$$\begin{aligned}
x_1 - \xi_1 x_3 &= Q^1 = \frac{R_{22}(L_1 + qR_1) - R_{12}(L_2 + qR_2)}{R_{22}R_{11} - R_{12}R_{21}}, \\
q &= \frac{L(R_{22}R_{11} - R_{12}R_{21}) - R_{22}R_1L_1 - R_{12}R_1L_2 + R_{21}R_2L_1 + R_{11}R_1L_1}{R_{22}R_1R_1 - R_{12}R_1R_2 + R_{21}R_2L_1 + R_{11}R_1L_1}, \\
x_2 - \xi_2 x_3 &= Q^2 = \frac{-R_{12}(L_1 + qR_1) + R_{22}(L_2 + qR_2)}{R_{22}R_{11} - R_{12}R_{21}},
\end{aligned} \tag{4.4}$$

where $L_1 \equiv \dfrac{\partial L}{\partial \xi_1}$ and so on. Compare with the analogous construction in the case of the Monge–Ampère equation [2].

5 General solution from the point of view of the Legendre Transformation

The Legendre transformation was employed in [4] to linearize the Universal equation. This transformation which is clearly involutive has the flavour of a twistor transformation. In a multivariable generalization it runs as follows [6].

Introduce a dual space with coordinates ξ_i, $i = 1, \ldots, d$ and a function $w(\xi_i)$ defined by

$$\phi(x_1, x_2, \ldots, x_d) + w(\xi_1, \xi_2, \ldots, \xi_d) = x_1\xi_1 + x_2\xi_2 + \ldots, x_d\xi_d,$$

(5.1)

$$\xi_i = \frac{\partial \phi}{\partial x_i}, \qquad x_i = \frac{\partial w}{\partial \xi_i}.$$

To evaluate the second derivatives ϕ_{ij} in terms of derivatives of w it is convenient to introduce two Hessian matrices Φ, W with matrix elements ϕ_{ij} and $w_{\xi_i\xi_j} = w_{ij}$, respectively. Then assuming that Φ is invertible, $\Phi W = 1$ and

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = (W^{-1})_{ij}, \qquad \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} = (\Phi^{-1})_{ij}.$$

(5.2)

The effect of the Legendre transformation upon the equation (0.01) is immediate; in the new variables the equation becomes simply

$$\sum_{i,j} \xi_i \xi_j \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} = 0,$$

(5.3)

a linear second order equation for w. The general solution of this equation is immediate: $w = F_1(\xi_1 \ldots \xi_d) + F_2(\xi_1 \ldots \xi_d)$ where F_1 is an arbitrary homogeneous function of weight zero and F_2 is an arbitrary homogeneous function of weight one. Then the general solution of the Universal equation is given implicitly as the eliminant of the variables ξ_i from the equations

$$\phi(x_1, x_2, \ldots, x_d) = \sum x_j \xi_j - w(\xi_1, \xi_2, \ldots, \xi_d), \qquad x_i = \frac{\partial w}{\partial \xi_i}.$$

(5.4)

6 The degenerate solutions of the previous section

The method of the previous section does not work if $\det(\phi_{ij}) = 0$. This means that the function ϕ satisfies the Monge–Ampère equation of n-dimensional space. As a corollary, there exists a linear dependence $\sum \phi_{ij}\alpha_j = 0$ among the columns of the matrix ϕ_{ij}. Let us multiply the columns of the matrix of the Universal equation by the corresponding multipliers α and sum them according to columns, say with index j. The result is a column with zeroes everywhere except for the first component. Let us repeat the same operation with the rows of the matrix. As a result we shall obtain

$$\det_{n+1}(\text{Universal matrix}) = \frac{1}{\alpha_i \alpha_j} R^2 \det_{n-1}(\tilde{\phi}_{ij}) = 0,$$

(6.1)

where the $(n-1) \times (n-1)$-matrix $\tilde{\phi}$ arises from the matrix ϕ by omitting its j-th column and i-th row.

So if ϕ is the solution of the Universal Equation and simultaneously $\det(\phi_{i,j}) = 0$ then all minors of $n-1$ order of the matrix ϕ are also zero and this corresponds to a degenerate solution of the Monge–Ampère equation [2].

7   Concluding remarks

The general solution of the Universal Equation described in this paper was obtained in just a few steps. We used only the algebraic operations of finding a common solution to a definite set of equations and no more.

However, the final form of the solution given in (3.1) has little algebro-geometrical character. We have observed that the solution of the Universal Equation is closely connected with the hydrodynamic equations (0.01), the solution of which may be expressed in terms of an algebraic curve (2.3), the general solution of which in its turn was reduced to (3.1).

The question arises as to whether this construction can be described only on the level of properties of the group of inner symmetry (discrete or continuous) of the Universal Equation.

Acknowledgement

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