Nonlinear Mathematical Physics 1995, V.2, N 1, 73–89.

Symmetry Reduction for Equation $\Box u + \lambda(u_1^2 + u_2^2 + u_3^2)^{1/2} u_0 = 0$

L.F. BARANNYK and H.O. LAHNO

Department of Mathematics, Pedagogical University, Ostrogradsky Street 2, 314003, Poltava, Ukraine

Submitted by A. NIKITIN
Received October 3, 1994

Abstract

The subalgebras of the invariance algebra of equation $\Box u+\lambda(u_1^2+u_2^2+u_3^2)^{1/2}u_0 = 0$ are described with respect to the conjugation. Reduction of this equation to a differential equation in a less number of independent variables is implemented by means of every subalgebra from the obtained list.

1 Introduction

The wide class of nonlinear wave equations which are invariant under the Euclidean groups are described in [1]. One such equation has the form [1]

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} + \lambda \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 \right]^{1/2} \frac{\partial u}{\partial x_0} = 0. \qquad (1)$$

This equation is invariant under the Lie algebra F generated by the following vector fields

$$P_0 = \frac{\partial}{\partial x_0}, \qquad P_a = \frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a},$$

$$D = x_0 \frac{\partial}{\partial x_0} + x^a \frac{\partial}{\partial x_a}, \qquad Z = \frac{\partial}{\partial u} \qquad (a < b; \quad a, b = 1, 2, 3).$$

Fushchych and Serova [1] have investigated the symmetry reduction of equation (1) with one-dimensional subalgebras of the algebra F and have found some exact solutions of this equation. These results have also been presented in the book [2].

Equation (1) is invariant under the mapping $(x_0, x_1, x_2, x_3, u) \to (x_0, -x_1, x_2, x_3, u)$. By carrying out the symmetry reduction we can consider subalgebras of the algebra F with respect to the conjugation defined by the group G, generated both by inner automorphisms of the algebra F and the automorphism

$$P_0 \to P_0, \quad P_1 \to -P_1, \quad P_2 \to P_2, \quad P_3 \to P_3, \quad D \to D,$$

$$Z \to Z, \quad J_{12} \to -J_{12}, \quad J_{13} \to -J_{13}, \quad J_{23} \to J_{23}.$$

Here the complete list of subalgebras of the algebra F are obtained with respect to G-conjugation and symmetry reductions of equation (1) are done on all subalgebras from this list. Also, new invariant solutions of this equation are found. Concepts and results of group analysis of differential equations can be found in the books of Ovsyannikov [3] and Olver [4].

Let L be a subalgebra of the algebra F and the rank of L be equal to r, $1 \leq r \leq 3$, with $k = 4 - r$. We shall designate by $\omega_1, \ldots, \omega_k, \omega_{k+1}$ a complete system of functionally independent invariants of the subalgebra L. Here we assume that the invariant ω_{k+1} depends on u and other invariants do not depend on u. Notation $L =< X_1, \ldots, X_s >$ denotes that $X_1, \ldots, X_s$ are generators of the algebra L. We shall designate the sequence of algebras $V_1 \dotplus K, \ldots, V_m \dotplus K$ by $K : V_1, \ldots, V_m$.

An ansatz corresponding to the subalgebra L has the form

$$\omega_{k+1} = \varphi(\omega_1, \ldots, \omega_k). \tag{2}$$

Let

$$\varphi_a = \frac{\partial \varphi}{\partial \omega_a}, \qquad \varphi_{ab} = \frac{\partial^2 \varphi}{\partial \omega_a \partial \omega_b}.$$

For $k = 1$ we shall write ω instead of ω_1, $\dot{\varphi}$ — instead of φ_1 and $\ddot{\varphi}$ — instead of φ_{11}.

For any given subalgebra of the algebra F after colon we point out the corresponding ansatz (2), solved with respect to u in terms of invariants $\omega_1, \ldots, \omega_k$, as well as the reduced equation received by means of the given ansatz from equation (1). We shall presuppose that $\lambda \neq 0$.

2 Classification of subalgebras of the invariance algebra

Symmetry reductions are realizable only when a subalgebra has an invariant which dependent on u. In connection with that one should consider only those subalgebras of the algebra F, which do not contain Z. As a representative of the class of subalgebras, which have with the respect to G-conjugation the same invariants, we shall take the subalgebra, which is not contained in any other subalgebra with this property. We call this subalgebra the I-maximal one. It is defined uniquely with respect to G-conjugation.

For description of subalgebras of the algebra F we use the general method suggested by Patera, Winternitz and Zassenhaus [5] and supplemented by series of structural propositions in the book of Fushchych et al. [6].

Using the Lie-Goursat classification method for subalgebras of algebraic sums of Lie algebras [5, 6], we obtain that non-zero subalgebras of the algebra $AO(3) \oplus < D, Z >$ are exhausted with respect to inner automorphisms by the following subalgebras:

$$< D + \alpha Z > (\alpha \in R); \ < Z >; \ < D, Z >;$$

$$< J_{12} + \alpha D + \beta Z >; \quad < J_{12} + \alpha Z, D + \beta Z >; \quad < J_{12}, Z >;$$

$$< J_{12}, D, Z >, \quad \alpha, \beta \in R; \tag{3}$$

$$AO(3); AO(3) \oplus < D + \alpha Z >; AO(3) \oplus < Z >; AO(3) \oplus < D, Z > .$$

Let K be one of the subalgebras (3) and $\widehat{K}$ be such a subalgebra of the algebra F that its projection onto $AO(3) \oplus < D, Z >$ coincides with K. If the projection of K onto $< D >$ is non-zero, K annules the only zero subspace of the space $U =< P_0, P_1, P_2, P_3 >$. Let us assume that $Z \notin K$. Then K could be considered as a completely reducible algebra of linear transformation of the space U. In accordance with theorem 1.5.3 [6], algebra $\widehat{K}$ is conjugated with the algebra of the form $V +) K$, where $V \subset U$. If $Z \in K$, then $K =< Z > \oplus K_1$. In accordance with the above arguments, $\widehat{K}$ is conjugated with the algebra of the form $(V + < Z + X >) +) K_1$ where $V \subset U$ and $X \in U$. Using proposition 1.2.2 [6] we conclude that $X \in V$. Therefore it is possible to assume that $X = 0$.

If the projection K onto $< D >$ is zero, then $\widehat{K}$ is a subalgebra of the direct sum $AE(3) \oplus < P_0, Z >$. Non-zero subalgebras of the algebra $AO(3) \oplus < P_0, Z >$ are exhausted with respect to inner automorphisms by subalgebras, which can be obtained as a result of the formal substitution D onto P_0 in the subalgebras (3). To classify subalgebras of the algebra $< P_1, P_2, P_3 > \oplus < P_0, Z >$, one should use the Lie-Goursat classification method and Witt's mapping theorem [7]. Let $\pi(K)$ be the projection K onto $AO(3)$. If $\pi(K) =< J_{12} >$, then because of the theorems 1.5.3 and III.4.1 [6] algebra $\widehat{K}$ contains, with respect to conjugation, its own projection onto $< P_1, P_2 >$. If $\pi(K) = AO(3)$ then its own $AO(3) \subset \widehat{K}$ and $\widehat{K}$ contains its own projection onto $< P_1, P_2, P_3 >$.

According to what has been said, it is not difficult to see that non-zero I-maximal subalgebras of the algebra F are exhausted with respect to G-conjugation by subalgebras:

A. Subalgebras having zero projections onto $AO(3)$:

$$< P_0 >, < \alpha P_0 + P_1 >, < P_0, P_1 >, < \beta P_0 + P_1, P_2 >,$$

where $\alpha > 0, \beta > 0$;

$$< D + \alpha Z >: 0, < P_0 >, < \beta P_0 + P_1 >, < P_0, P_1 >, < \gamma P_0 + P_1, P_2 >,$$

where $\alpha, \beta, \gamma \in R$ and $\beta \geq 0, \gamma > 0$;

$$< Z \pm P_0 >: 0, < \alpha P_0 + P_1 >, < \beta P_0 + P_1, P_3 >,$$

where $\alpha > 0, \beta \geq 0$;

$$< Z + \alpha P_0 + P_2 >: 0, < \beta P_0 + P_1 >, < \gamma P_0 + P_1, P_3 >,$$

where $\alpha, \beta, \gamma \in R$ and $\beta \geq 0, \gamma > 0$;

$$< Z + P_1 >:< P_0 >, < P_0, P_3 > .$$

B. Subalgebras having zero projections onto $< D, Z >$ and having non-zero projections onto $AO(3)$:

$$< J_{12} >: 0, < P_0 >, < \alpha P_0 + P_3 >, < P_0, P_3 >, < P_1, P_2 >,$$

$< P_0, P_1, P_2 >, < \beta P_0 + P_3, P_1, P_2 >,$

where $\alpha \geq 0, \beta > 0;$

$< J_{12} + P_0 >: 0, < \alpha P_0 + P_3 > (\alpha \geq 0);$

$< J_{12} + \alpha P_0 + P_3 > (\alpha \geq 0); < J_{12} + P_3, P_0 >;$

$< J_{12}, J_{13}, J_{23} >: 0, < P_0 >, < P_1, P_2, P_3 >, < P_0, P_1, P_2, P_3 > .$

C. Subalgebras having zero projections onto $< D >$ and having non-zero projections onto $AO(3)$ and $< Z >$:

$< J_{12} + \alpha Z >: 0, < P_0 >, < \beta P_0 + P_3 >, < P_0, P_3 >,$

where $\alpha > 0, \beta \geq 0;$

$< J_{12} + P_0 + \alpha Z >: 0, < \beta P_0 + P_3 >,$

where $\alpha \neq 0,\ \beta \geq 0;$

$< J_{12} + \alpha P_0 + P_3 + \beta Z > \quad (\alpha \in R, \beta > 0);$

$< J_{12} + P_3 + \alpha Z, P_0 > \ (\alpha > 0);$

$< J_{12} >: < P_0 + \alpha Z, P_1, P_2 >, < \beta P_0 + P_3 + Z, P_1, P_2 >,$

$< P_0 + \alpha Z, P_3 + \gamma P_0, P_1, P_2 >, < P_0, P_3 + Z, P_1, P_2 >,$

where $\alpha = \pm 1, \beta \in R, \gamma > 0;$

$< P_0 + \alpha Z, J_{12} + \beta P_0 + \gamma P_3 >,$

where $\alpha = \pm 1, \beta > 0, \gamma \in R$ or $\alpha = \pm 1, \beta = 0, \gamma \geq 0;$

$< \alpha P_0 + P_3 + Z, J_{12} + \beta P_0 + \gamma P_3 >,$

where $\alpha, \beta, \gamma \in R$ and $\beta > 0$ or $\beta = 0, \gamma \geq 0;$

$< P_0 + \alpha Z, P_3 + \beta P_0, J_{12} + \gamma P_0 >,$

where $\alpha = \pm 1, \beta \geq 0, \gamma \geq 0;$

$< P_0, P_3 + Z, J_{12} + \alpha P_3 > (\alpha \geq 0);$

$< J_{12}, J_{13}, J_{23} >: < P_0 + \alpha Z >, < P_0 + \alpha Z, P_1, P_2, P_3 > \quad (\alpha = \pm 1).$

D. Subalgebras having non-zero projections onto $AO(3)$ and $< D >$:

$< J_{12} + \alpha D + \beta Z >: 0, < P_0 >, < \gamma P_0 + P_3 >, < P_0, P_3 >,$

where $\alpha, \beta, \gamma \in R$ and $\alpha > 0, \quad \gamma \geq 0;$

$< J_{12} + \alpha Z, D + \beta Z >: 0, < P_0 >, < \gamma P_0 + P_3 >, < P_0, P_3 >,$

where $\alpha, \beta, \gamma \in R$ and $\alpha \geq 0, \quad \gamma \geq 0$;

$< J_{12}, D + \alpha Z > +) < P_1, P_2 >, (\alpha \in R)$;

$< J_{12}, D + \alpha Z >:< P_0, P_1, P_2 >, < \beta P_0 + P_3, P_1, P_2 >$,

where $\alpha \neq 0, \beta > 0$;

$< J_{12}, J_{13}, J_{23}, D + \alpha Z > (\alpha \in R)$;

$< J_{12}, J_{13}, J_{23}, D + \alpha Z >:< P_0 >, < P_1, P_2, P_3 > (\alpha \neq 0)$;

$< J_{12}, J_{13}, J_{23}, D, P_0, P_1, P_2, P_3 >$.

We note that the subalgebras of maximal rank three are used for our reduction.

3 The reduction of (1) to ordinary differential equations

3.1. $< P_0, P_3, D + \alpha Z > (\alpha \in R) : u = \frac{\alpha}{2} \ln\{x_1^2 + x_2^2\} + \varphi(\omega),\ \omega = \frac{x_1}{x_2}$,

$$(1 + \omega^2)\ddot{\varphi} + 2\omega\dot{\varphi} = 0.$$

In this case $\varphi = C_1 \arctan\omega + C_2$ and therefore

$$u = \frac{\alpha}{2} \ln\{x_1^2 + x_2^2\} + C_1 \arctan\frac{x_1}{x_2} + C_2,$$

where C_1 and C_2 are arbitrary constants.

3.2. $< \alpha P_0 + P_1, P_3, D + \beta Z > \ (\alpha > 0, \quad \beta \in R) :$

$$u = \beta \ln x_2 + \varphi(\omega),\ \omega = \frac{x_2}{x_0 - \alpha x_1},$$

$$[(1 - \alpha^2)\omega^2 - 1]\ddot{\varphi} + 2(1 - \alpha)\omega\dot{\varphi} -$$

$$\varepsilon\lambda\omega\sqrt{\alpha^2\omega^2\dot{\varphi}^2 + (\beta\omega^{-1} + \dot{\varphi})^2}\dot{\varphi} + \beta\omega^{-2} = 0, \tag{4}$$

where $\varepsilon = \text{sign}\ (x_0 - \alpha_1 x)$.

Let $\beta = 0$. Then equation (4) is of the form

$$[(1 - \alpha^2)\omega^2 - 1]\ddot{\varphi} + 2(1 - \alpha)\omega\dot{\varphi} - \varepsilon\lambda\omega\sqrt{\alpha^2\omega^2 + 1}|\dot{\varphi}|\dot{\varphi} = 0. \tag{5}$$

For $\alpha = 1$ and $\dot{\varphi} > 0$ we obtain that

$$\dot{\varphi} = \frac{3}{\varepsilon\lambda(\omega^2 + 1)^{\frac{3}{2}} + \widetilde{C}}.$$

If $\widetilde{C} = 0$ then

$$\varphi = \frac{3}{\varepsilon\lambda(\omega^2 + 1)^{\frac{1}{2}}} + C.$$

The function

$$u = \frac{3}{\lambda} \frac{x_2}{\sqrt{(x_0 - x_1)^2 + x_2^2}} + C,$$

where $x_0 - x_1 > 0$ for $\lambda > 0$ and $x_0 - x_1 < 0$ for $\lambda < 0$ is the corresponding invariant solution of (1).

For $\alpha = 1$ and $\dot{\varphi} < 0$ we receive the solution

$$u = -\frac{3}{\lambda} \frac{x_2}{\sqrt{(x_0 - x_1)^2 + x_2^2}} + C,$$

where $\lambda(x_0 - x_1) > 0$.

Let $\alpha \neq 1$ and $\varphi > 0$. The substitution $\psi = 1/\dot{\varphi}$ transforms (5) into the linear equation

$$[(1-\alpha^2)\omega^2 - 1]\dot{\psi} - 2(1-\alpha^2)\omega\psi + \varepsilon\lambda\omega\sqrt{\alpha^2\omega^2 + 1} = 0. \tag{6}$$

If

$$\frac{1}{1-\alpha^2} = \rho^2,$$

then the function

$$\psi = \frac{\lambda\alpha^2\varepsilon}{(1-\alpha^2)^2}[1 + (\alpha^2 - 1)\omega^2] \left\{ \frac{1}{4\rho} \ln \left| \frac{\sqrt{\alpha^2\omega^2 + 1} - \rho}{\sqrt{\alpha^2\omega^2 + 1} + \rho} \right| - \frac{\sqrt{\alpha^2\omega^2 + 1}}{2(\alpha^2\omega^2 + 1 - \rho^2)} + C_1 \right\}$$

is the general solution of (6). If

$$\frac{1}{1-\alpha^2} = -\rho^2,$$

then the general solution of (6) is

$$\psi = \frac{\lambda\alpha^2\varepsilon}{(1-\alpha^2)^2}[1 + (\alpha^2 - 1)\omega^2] \left\{ -\frac{\sqrt{\alpha^2\omega^2 + 1}}{2(\alpha^2\omega^2 + 1 + \rho^2)} + \frac{1}{2\rho} \arctan \frac{\sqrt{\alpha^2\omega^2 + 1}}{\rho} + C_1 \right\}.$$

In every of the cases the solution of (1) is of the form

$$\varphi = \int \frac{d\omega}{\psi} + C_2, \quad \text{and} \quad u = \varphi(\omega).$$

3.3. $< \alpha P_0 + P_1, P_3, Z + \beta P_0 > \quad (\alpha > 0, \quad \beta = \pm 1):$

$$u = \beta x_0 - \alpha\beta x_1 + \varphi(\omega), \quad \omega = x_2,$$

$$\ddot{\varphi} - \beta\lambda\sqrt{\alpha^2 + \dot{\varphi}^2} = 0.$$

It is easy to find that

$$\dot{\varphi} = \frac{C^2 e^{2\lambda\beta\omega} - \alpha^2}{2Ce^{\lambda\beta\omega}} \qquad (C \neq 0)$$

and therefore

$$\varphi = C_1 e^{\lambda\beta\omega} + \frac{\alpha^2}{4\lambda^2 C_1} e^{-\lambda\beta\omega} + C_2,$$

$$u = \beta x_0 - \alpha\beta x_1 + C_1 e^{\lambda\beta x_2} + \frac{\alpha^2}{4\lambda^2 C_1} e^{-\lambda\beta x_2} + C_2,\ C_1 \neq 0.$$

3.4. $< P_0, P_3, Z + P_1 >$: $\quad u = x_1 + \varphi(\omega), \quad \omega = x_2, \quad \ddot{\varphi} = 0.$

The corresponding invariant solution of (1) is of the form $u = x_1 + C_1 x_1 + C_2$, where C_1 and C_2 are arbitrary constants.

3.5. $< \alpha P_0 + P_1, P_3, Z + P_2 > (\alpha > 0): \ u = x_2 + \varphi(\omega),\ \omega = x_0 - \alpha x_1,$

$$(1 - \alpha^2)\ddot{\varphi} + \lambda\dot{\varphi}\sqrt{\alpha^2\dot{\varphi}^2 + 1)} = 0.$$

If $\alpha = 1$ then $\dot{\varphi} = 0$ and therefore the corresponding invariant solution of (1) is of the form $u = x_2 + C$.

If $\alpha \neq 1$ then

$$\varphi = \frac{\alpha^2 - 1}{\alpha\lambda} \ln\left|\frac{1 + C_1 \exp(\frac{\lambda}{\alpha^2 - 1}\omega)}{1 - C_1 \exp(\frac{\lambda}{\alpha^2 - 1}\omega)}\right| + C_2,$$

where $C_1 > 0$. The corresponding invariant solution of (1) is

$$u = x_2 + \frac{\alpha^2 - 1}{\alpha\lambda} \ln\left|\frac{1 + C_1 \exp(\frac{\lambda}{\alpha^2 - 1}(x_0 - \alpha x_1))}{1 - C_1 \exp(\frac{\lambda}{\alpha^2 - 1}(x_0 - \alpha x_1))}\right| + C_2.$$

3.6. $< \alpha P_0 + P_1, P_3, Z + \beta P_0 + P_2 > (\alpha > 0, \beta \neq 0):$

$$u = x_2 + \varphi(\omega), \quad \omega = x_0 - \alpha x_1 - \beta x_2,$$

$$(1 - \alpha^2 - \beta^2)\ddot{\varphi} + \lambda\dot{\varphi}\sqrt{\alpha^2\dot{\varphi}^2 + (1 - \beta\dot{\varphi})^2} = 0.$$

In this case the corresponding solution of (1) is of the form

$$u = x_2 + \frac{\beta^2 + \alpha^2 - 1}{\lambda\sqrt{\alpha^2 + \beta^2}} \times$$

$$\ln \frac{C_1 \exp\{\frac{\lambda(x_0 - \alpha x_1 - \beta x_2)}{1 - \alpha^2 - \beta^2}\}(C^2(\alpha^2 + \beta^2)^{\frac{3}{2}} - \beta) - (C^2(\alpha^2 + \beta^2)^{\frac{3}{2}} + \beta)}{C(\alpha^2 + \beta^2)(C_1 \exp\{\frac{\lambda\omega}{1 - \alpha^2 - \beta^2}\} + 1)},$$

where $C \neq 0, \quad C_1 \neq 0, \quad 1 - \alpha^2 - \beta^2 \neq 0.$

3.7. $< \alpha P_0 + P_1, P_2, P_3, \sigma(\alpha) J_{12}, \sigma(\alpha) J_{13}, J_{23} >,$

where $\alpha \geq 0,$

$$\sigma(\alpha) = 1 \ \text{ for } \ \alpha = 0 \ \text{ and } \ \sigma(\alpha) = 0 \ \text{ for } \ \alpha \neq 0:$$

$$u = \varphi(\omega), \quad \omega = x_0 - \alpha x_1,$$

$$(1-\alpha^2)\ddot{\varphi} + \lambda\alpha|\dot{\varphi}|\dot{\varphi} = 0.$$

From the solutions of this reduced equation we obtain the following solutions of (1)

$$u = C_1 x_0 + C_2 \quad \text{for} \quad \alpha = 0;$$

$$u = \frac{1-\alpha^2}{\alpha\lambda}\ln[\alpha\lambda(x_0 - \alpha x_1) + C_1] + C_2 \quad \text{for} \quad 0 < \alpha \le 1;$$

$$u = \frac{1-\alpha^2}{\alpha\lambda}\ln[\alpha\lambda(\alpha x_1 - x_0) + C_1] + C_2 \quad \text{for} \quad \alpha > 1.$$

3.8. $< J_{12} + \alpha Z, P_0 + \beta J_{12}, P_3 + \gamma J_{12} > (\alpha > 0, \beta \in R, \gamma \ge 0):$

$$u = \alpha \arctan\frac{x_2}{x_1} - \alpha\beta x_0 - \alpha\gamma x_3 + \varphi(\omega), \quad \omega = x_1^2 + x_2^2,$$

$$4\omega\ddot{\varphi} + 4\dot{\varphi} + \alpha\beta\lambda\sqrt{4\omega\dot{\varphi}^2 + \alpha^2\omega^{-1} + \alpha^2\gamma^2} = 0.$$

For $\beta = 0$ we found that $\varphi = C_1 \ln\omega + C_2$. The corresponding solution of (1) is

$$u = \alpha \arctan\frac{x_2}{x_1} - \alpha\gamma x_3 + C_1 \ln\{x_1^2 + x_2^2\} + C_2.$$

3.9. $< P_0 + \alpha Z, P_3 + \beta P_0, J_{12} >$, where $\alpha = 0, \pm 1, \beta \ge 0:$

$$u = \alpha x_0 - \alpha\beta x_3 + \varphi(\omega), \quad \omega = x_1^2 + x_2^2,$$

$$4\omega\ddot{\varphi} + 4\dot{\varphi} - \alpha\lambda\sqrt{4\omega\dot{\varphi}^2 + \alpha^2\beta^2} = 0.$$

If $\beta = 0, \ \dot{\varphi} > 0$ then the reduced equation is of the form

$$2\omega\ddot{\varphi} + (2 - \alpha\lambda\sqrt{\omega})\dot{\varphi} = 0.$$

As far as

$$\dot{\varphi} = C_1\omega^{-1}e^{\alpha\lambda\sqrt{\omega}},$$

then for $\alpha \ne 0$ equation (1) has the solution

$$u = \alpha x_0 + C_1 \int \frac{e^y}{y} dy + C_2, \quad \text{where} \quad y = \alpha\lambda\sqrt{x_1^2 + x_2^2}, \quad C_1 > 0.$$

For $\beta = 0, \dot{\varphi} < 0$ we found an analogous solution

$$u = \alpha x_0 + C_1 \int \frac{e^y}{y} dy + C_2, \quad \text{where} \quad y = -\alpha\lambda\sqrt{x_1^2 + x_2^2}, \quad C_1 > 0.$$

For $\alpha = 0$ the function $u = C_1 \ln\{x_1^2 + x_2^2\} + C_2$ is the solution of (1).

3.10 $< P_0, P_3 + Z, J_{12} >$: $\quad u = x_3 + \varphi(\omega), \quad \omega = x_1^2 + x_2^2,$

$$\omega\ddot{\varphi} + \dot{\varphi} = 0.$$

It is easy to find that $u=x_3+C_1\ln(x_1^2+x_2^2)+C_2$.

3.11. $< P_0+\alpha Z, P_2, P_3, J_{23} > (\alpha=0,\pm1):\quad u=\alpha x_0+\varphi(\omega),\quad \omega=x_1,$

$$\ddot{\varphi}-\lambda\alpha|\dot{\varphi}|=0.$$

The corresponding invariant solution is the function

$$u=\alpha x_0+C_1e^{\pm\alpha\lambda x_1}+C_2,$$

where $\alpha\lambda C_1\geq 0$.

3.12. $< \alpha P_0+P_1+Z, P_2, P_3, J_{23} > (\alpha\in R):\ u=x_1+\varphi(\omega),\ \omega=x_0-\alpha x_1,$

$$(1-\alpha)\ddot{\varphi}+\lambda|1-\alpha\dot{\varphi}|\dot{\varphi}=0.$$

If $\alpha=\pm1$ then $\varphi=\pm\omega+C$ and therefore $u=\pm x_0+C$. Let $\alpha\neq 0$ and $\alpha\neq\pm1$. For $1-\alpha\dot{\varphi}>0$ we receive

$$\varphi=\frac{\alpha^2-1}{\alpha\lambda}\ln\left[1+\alpha C_1\exp\left(\frac{\lambda}{\alpha^2-1}\omega\right)\right]+C_2,$$

and for $1-\alpha\dot{\varphi}<0$

$$\varphi=\frac{1-\alpha^2}{\alpha\lambda}\ln\left[1+\alpha C_1\exp\left(\frac{\lambda}{1-\alpha^2}\omega\right)\right]+C_2.$$

Hence equation (1.1) has the invariant solutions

$$u=x_1+\frac{\alpha^2-1}{\alpha\lambda}\ln\left[1+\alpha C_1\exp\left(\frac{\lambda}{\alpha^2-1}(x_0-\alpha x_1)\right)\right]+C_2,$$

$$u=x_1+\frac{1-\alpha^2}{\alpha\lambda}\ln\left[-1-\alpha C_1\exp\left(\frac{\lambda}{\alpha^2-1}(x_0-\alpha x_1)\right)\right]+C_2.$$

The values of parameters and range of values x_0,x_1 are defined by positive expressions under the logarithm sign.

For $\alpha=0$ we obtain the invariant solution

$$u=x_1+C_1e^{-\lambda x_0}+C_2.$$

3.13. $< J_{12}, J_{13}, J_{23} > \oplus < P_0+\alpha Z > (\alpha=0,\pm1):$

$$u=\alpha x_0+\varphi(\omega),\quad \omega=\sqrt{x_1^2+x_2^2+x_3^2},$$

$$\ddot{\varphi}+\dot{\varphi}-\lambda\alpha|\dot{\varphi}|=0.$$

For $\dot{\varphi}>0$ we obtain

$$\varphi=C_1\int\frac{e^{\alpha\lambda\omega}}{\omega^2}d\omega+C_2,\quad C_1>0.$$

The corresponding invariant solution of (1) is of the form

$$u=\alpha x_0+\alpha\lambda C_1\left(-\frac{e^y}{y}+\int\frac{e^y}{y}dy\right)+C_2,\quad C_1>0,$$

where $y = \alpha\lambda\sqrt{x_1^2 + x_2^2 + x_3^2}$. For $\dot{\varphi} < 0$ we obtain the invariant solution

$$u = \alpha x_0 + \alpha\lambda C_1 \left(-\frac{e^y}{y} + \int \frac{e^y}{y} dy \right) + C_2, \qquad C_1 > 0,$$

where $y = -\alpha\lambda\sqrt{x_1^2 + x_2^2 + x_3^2}$.

3.14. $< P_0, P_3, J_{12} + \alpha D + \beta Z > (\alpha > 0, \beta \in R)$:

$$u = \beta \arctan \frac{x_2}{x_1} + \varphi(\omega), \quad \omega = 2\alpha \arctan \frac{x_2}{x_1} - \ln(x_1^2 + x_2^2),$$

$$\ddot{\varphi} = 0.$$

In this case $\varphi = C_1\omega + C_2$ and therefore

$$u = \beta \arctan \frac{x_2}{x_1} + C_1 \left(2\alpha \arctan \frac{x_2}{x_1} - \ln(x_1^2 + x_2^2) \right) + C_2.$$

3.15. $< P_0, D + \alpha J_{12}, J_{12} + \beta Z > \quad (\alpha \in R, \beta > 0)$:

$$u = -\frac{\alpha\beta}{2} \ln\{x_1^2 + x_2^2\} + \beta \arctan \frac{x_2}{x_1} + \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2}{x_3^2},$$

$$2\omega(\omega + 1)\ddot{\varphi} + (3\omega + 2)\dot{\varphi} = 0.$$

In this case we receive following invariant solution of (1):

$$u = -\frac{\alpha\beta}{2} \ln\{x_1^2 + x_2^2\} + \beta \arctan \frac{x_2}{x_1} + C_1 \ln \left| \frac{\sqrt{x_1^2 + x_2^2 + x_3^2} - |x_3|}{\sqrt{x_1^2 + x_2^2 + x_3^2} - |x_3|} \right| + C_2.$$

3.16. $< P_0, J_{12}, D + \alpha Z > \quad (\alpha \in R)$:

$$u = \frac{\alpha}{2} \ln\{x_1^2 + x_2^2\} + \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2}{x_3^2},$$

$$2\omega(\omega + 1)\ddot{\varphi} + (3\omega + 2)\dot{\varphi} = 0.$$

The corresponding invariant solution of (1) is of the form

$$u = \frac{\alpha}{2} \ln\{x_1^2 + x_2^2\} + C_1 \ln \left| \frac{\sqrt{x_1^2 + x_2^2 + x_3^2} - |x_3|}{\sqrt{x_1^2 + x_2^2 + x_3^2} + |x_3|} \right| + C_2.$$

3.17. $< \gamma P_0 + P_3, D + \alpha J_{12}, Z + \beta^{-1} J_{12} > \ (\alpha \in R, \beta > 0, \gamma \geq 0)$:

$$u = \beta \arctan \frac{x_2}{x_1} - \frac{\alpha\beta}{2} \ln\{x_1^2 + x_2^2\} + \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2}{(x_0 - \gamma x_3)^2},$$

$$4\omega((1 - \gamma^2)\omega - 1)\ddot{\varphi} + (6(1 - \gamma^2)\omega - 4)\dot{\varphi} -$$

$$2\varepsilon\lambda\omega\dot{\varphi}\sqrt{4\omega(1 + \gamma^2\omega)\dot{\varphi}^2 - 4\alpha\lambda\dot{\varphi} + \beta^2(1 + \alpha^2)\omega^{-1}} = 0,$$

where $\varepsilon = \text{sign }(x_0 - \gamma x_3)$.

3.18. $<\beta P_0 + P_3, J_{12}, D + \alpha Z>$ $(\alpha \in R, \beta \geq 0):$

$$u = \frac{\alpha}{2}\ln\{x_1^2 + x_2^2\} + \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2}{(x_0 - \beta x_3)^2},$$

$$(4(1-\beta^2)\omega^2 - 4\omega)\ddot{\varphi} + (6(1-\beta^2)\omega - 4)\dot{\varphi} -$$

$$2\varepsilon\omega\lambda\dot{\varphi}\sqrt{4\omega(1+\beta^2\omega)\dot{\varphi}^2 + 4\alpha\dot{\varphi} + \alpha^2\omega^{-1}} = 0,$$

where $\varepsilon = \text{sign }(x_0 - \beta x_3)$.

3.19. $<P_2, P_3, J_{23}, D + \alpha Z>$ $(\alpha \in R):$

$$u = \alpha \ln x_0 + \varphi(\omega), \qquad \omega = \frac{x_1}{x_0},$$

$$(\omega^2 - 1)\ddot{\varphi} + 2\omega\dot{\varphi} + \lambda|\dot{\varphi}|(\alpha - \omega\dot{\varphi}) - \alpha = 0.$$

For $\alpha = 0$ and $\lambda > 0$ we obtain the solution

$$u = \frac{2x_1}{\lambda x_0} + C, \quad x_0 > 0.$$

3.20. $<J_{12}, J_{13}, J_{23}, D + \alpha Z>$ $(\alpha \in R):$

$$u = \alpha \ln x_0 + \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2 + x_3^2}{x_0^2},$$

$$4\omega(\omega - 1)\ddot{\varphi} + 6(\omega - 1)\dot{\varphi} + 2\lambda\sqrt{\omega}(\alpha - 2\omega\dot{\varphi})|\dot{\varphi}| - \alpha = 0.$$

For $\alpha = 0$ and $\dot{\varphi} > 0$ we find that

$$u = \varphi = \int \frac{\omega^{\frac{3}{2}}}{\lambda \ln|\frac{\omega}{\omega-1}| + C_1} d\omega + C_2.$$

4 Reduction of (1) to differential equations having two independent variables

4.1. $<P_0, P_3>$: $u = \varphi(\omega_1, \omega_2), \ \omega_1 = x_1, \ \omega_2 = x_2,$

$$\varphi_{11} + \varphi_{22} = 0.$$

4.2. $<\alpha P_0 + P_1, P_3>$ $(\alpha > 0):$ $u = \varphi(\omega_1, \ \omega_2), \ \omega_1 = x_0 - \alpha x_1, \ \omega_2 = x_2,$

$$(1 - \alpha^2)\varphi_{11} - \varphi_{22} + \lambda\varphi_1\sqrt{\alpha^2\varphi_1^2 + \varphi_2^2} = 0.$$

4.3. $<P_0, D + \alpha Z>$ $(\alpha \in R):$

$$u = \alpha \ln x_3 + \varphi(\omega_1, \omega_2), \quad \omega_1 = \frac{x_1}{x_3}, \quad \omega_2 = \frac{x_2}{x_3},$$

$$(1 + \omega_1^2)\varphi_{11} + (1 + \omega_2^2)\varphi_{22} + 2\omega_1\omega_2\varphi_{12} + 2\omega_1\varphi_1 + 2\omega_2\varphi_2 - \alpha = 0.$$

4.4. $< \alpha P_0 + P_1, D + \beta Z > (\alpha \geq 0, \beta \in R):$

$$u = \beta \ln x_3 + \varphi(\omega_1, \omega_2), \quad \omega_1 = \frac{x_0 - \alpha x_1}{x_3}, \quad \omega_2 = \frac{x_2}{x_3},$$

$$(1 - \alpha^2 - \omega_1^2)\varphi_{11} - (1 + \omega_2^2)\varphi_{22} - 2\omega_1\omega_2\varphi_{12} - 2\omega_1\varphi_1 -$$

$$2\omega_2\varphi_2 + \beta + \lambda\varphi_1\sqrt{\alpha^2\varphi_1^2 + \varphi_2^2 + (\beta - \omega_1\varphi_1 - \omega_2\varphi_2)^2} = 0.$$

4.5. $< \alpha P_0 + P_3, P_0 + \beta Z > (\alpha \geq 0, \beta = \pm 1):$

$$u = \beta x_0 - \alpha\beta x_3 + \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1, \quad \omega_2 = x_2,$$

$$\varphi_{11} + \varphi_{22} - \lambda\beta\sqrt{\alpha^2\beta^2 + \varphi_1^2 + \varphi_2^2} = 0.$$

4.6. $< \alpha P_0 + P_1, Z + \beta P_0 + P_2 > \quad (\alpha \geq 0, \quad \beta \in R):$

$$u = x_2 + \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 - \alpha x_1 - \beta x_2, \quad \omega_2 = x_3,$$

$$(1 - \alpha^2 - \beta^2)\varphi_{11} - \varphi_{22} + \lambda\varphi_1\sqrt{\alpha^2\varphi_1^2 + (1 - \beta\varphi_1)^2 + \varphi_2^2} = 0.$$

4.7. $< P_0, Z + P_3 >$: $\quad u = x_3 + \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1, \quad \omega_2 = x_2,$

$$\varphi_{11} + \varphi_{22} = 0.$$

4.8. $< P_0, J_{12} + \alpha D + \beta Z > (\alpha, \beta \in R):$

$$u = \beta \arctan \frac{x_2}{x_1} + \varphi(\omega_1, \omega_2), \quad \omega_1 = \frac{x_1^2 + x_2^2}{x_3^2},$$

$$\omega_2 = 2\alpha \arctan \frac{x_1}{x_2} + \ln(x_1^2 + x_2^2),$$

$$2\omega_1(1 + \omega_1)\varphi_{11} + 2(1 + \alpha^2)\omega_1^{-1}\varphi_{22} + 4\varphi_{12} + (2 + 3\omega_1)\varphi_1 = 0.$$

4.9. $< \alpha P_0 + P_3, J_{12} + \beta D + \gamma Z > \quad (\alpha \geq 0, \quad \beta \geq 0, \quad \gamma \in R):$

$$u = \gamma \arctan \frac{x_2}{x_1} + \varphi(\omega_1, \omega_2), \quad \omega_1 = \frac{x_1^2 + x_2^2}{(x_0 - \alpha x_3)^2},$$

$$\omega_2 = 2\beta \arctan \frac{x_2}{x_1} - \ln(x_1^2 + x_2^2),$$

$$4\omega_1((1 - \alpha^2)\omega_1 - 1)\varphi_{11} - 4(1 + \beta^2)\omega_1^{-1}\varphi_{22} +$$

$$8\varphi_{12} + (6(1 - \alpha^2)\omega_1 - 4)\varphi_1 - 2\varepsilon\lambda\omega_1\varphi_1 \times$$

$$\sqrt{4\omega_1(1+\alpha^2\omega_1)\varphi_1^2+4(1+\beta^2)\omega_1^{-1}\varphi_2^2-8\varphi_1\varphi_2+4\beta\gamma\omega_1^{-1}\varphi_2+\gamma^2\omega_1^{-1}}=0,$$

where $\varepsilon=$sign$(x_0-\alpha x_3)$.

4.10. $<P_2,P_3,J_{23}>$: $u=\varphi(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=x_1,$

$$\varphi_{11}-\varphi_{22}+\lambda\varphi_1|\varphi_2|=0.$$

4.11. $<\alpha P_0+P_3,J_{12}+P_0+\beta Z>\quad (\alpha\geq 0,\quad \beta\in R):$

$$u=\beta x_0-\alpha\beta x_3+\varphi(\omega_1,\omega_2),\quad \omega_1=x_0-\alpha x_3+\arctan\frac{x_1}{x_2},\quad \omega_2=x_1^2+x_2^2,$$

$$(1-\alpha^2-\omega_2^{-1})\varphi_{11}-4\omega_2\varphi_{22}-4\varphi_2+$$

$$\lambda\sqrt{(\alpha^2+\omega_2^{-1})\varphi_1^2+4\omega_2\varphi_2^2+2\alpha^2\beta\varphi_1+\alpha^2\beta^2}(\beta+\varphi_1)=0.$$

4.12. $<P_0,J_{12}+P_3+\alpha Z>(\alpha\geq 0):$

$$u=\alpha x_3+\varphi(\omega_1,\omega_2),\quad \omega_1=x_1^2+x_2^2,\quad \omega_2=\arctan\frac{x_1}{x_2}+x_3,$$

$$4\omega_1\varphi_{11}+(1+\omega_1^{-1})\varphi_{22}+4\varphi_1=0.$$

4.13. $<J_{12},J_{13},J_{23}>$: $u=\varphi(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=x_1^2+x_2^2+x_3^2,$

$$\varphi_{11}-4\omega_2\varphi_{22}-6\varphi_2+2\lambda\sqrt{\omega_2}\varphi_1|\varphi_2|=0.$$

4.14. $<P_0+\alpha Z,J_{12}+\beta P_0+\gamma P_3>\quad (\alpha=\pm 1,\quad \beta\geq 0,\quad \gamma\in R):$

$$u=\alpha x_0+\alpha\beta\arctan\frac{x_1}{x_2}+\varphi(\omega_1,\omega_2),$$

$$\omega_1=x_1^2+x_2^2,\quad \omega_2=x_3+\gamma\arctan\frac{x_1}{x_2},$$

$$4\omega_1\varphi_{11}+(1+\gamma^2\omega_1^{-1})\varphi_{22}+4\varphi_1-$$

$$\alpha\lambda\sqrt{4\omega_1\varphi_1^2+(1+\gamma^2\omega_1^{-1})\varphi_2^2+2\alpha\beta\gamma\omega_1^{-1}\varphi_2+\alpha^2\beta^2\omega_1^{-1}}=0.$$

4.15. $<\alpha P_0+P_3+Z,J_{12}+\beta P_0+\gamma P_3>$, where $\alpha,\beta,\gamma\in R$ and $\beta\geq 0:$

$$u=x_3+\gamma\arctan\frac{x_1}{x_2}+\varphi(\omega_1,\omega_2),$$

$$\omega_1=x_1^2+x_2^2,\quad \omega_2=x_0-\alpha x_3+(\beta-\alpha\gamma)\arctan\frac{x_1}{x_2},$$

$$4\omega_1\varphi_{11}+(\alpha^2-1+(\beta-\alpha\gamma)^2\omega_1^{-1})\varphi_{22}+4\varphi_1-$$

$$\lambda\varphi_2\sqrt{4\omega_1\varphi_1^2+(\alpha^2+(\beta-\alpha\gamma)^2\omega_1^{-1})\varphi_2^2+2(-\alpha+(\beta-\alpha\gamma)\gamma\omega_1^{-1})\varphi_2+1+\gamma^2\omega_1^{-1}}=0.$$

4.16. $< J_{12} + \alpha Z, D + \beta J_{12} >$ $(\alpha \geq 0, \quad \beta \in R)$:

$$u = \alpha \arctan \frac{x_2}{x_1} - \alpha\beta \ln x_0 + \varphi(\omega_1, \omega_2),$$

$$\omega_1 = \frac{x_1^2 + x_2^2}{x_0^2}, \quad \omega_2 = \frac{x_3}{x_0},$$

$$4\omega_1(\omega_1 - 1)\varphi_{11} + (\omega_2^2 - 1)\varphi_{22} + 4\omega_1\omega_2\varphi_{12} + (6\omega_1 - 4)\varphi_1 + 2\omega_2\varphi_2 + \alpha\beta -$$

$$\lambda\sqrt{4\omega_1\varphi_1^2 + \varphi_2^2 + \alpha^2\omega_1^{-1}}(2\omega_1\varphi_1 + \omega_2\varphi_2 + \alpha\beta) = 0.$$

4.17. $< J_{12}, D + \alpha Z >$ $(\alpha \neq 0)$: $u = \alpha \ln x_3 + \varphi(\omega_1, \omega_2)$,

$$\omega_1 = \frac{x_1^2 + x_2^2}{x_0^2}, \quad \omega_2 = \frac{x_3}{x_0},$$

$$4\omega_1(\omega_1 - 1)\varphi_{11} + (\omega_2^2 - 1)\varphi_{22} + 4\omega_1\omega_2\varphi_{12} + (6\omega_1 - 4)\varphi_1 + 2\omega_2\varphi_2 + \alpha\omega_2^{-2} -$$

$$\lambda\varepsilon\sqrt{4\omega_1\varphi_1^2 + \varphi_2^2 + 2\alpha\omega_2^{-1}\varphi_2 + \alpha^2\omega_2^{-2}}(2\omega_1\varphi_1 + \omega_2\varphi_2) = 0,$$

where $\varepsilon =$sign x_0.

5 Reduction of (1) to differential equations having three independent variables

5.1. $< P_0 >$: $u = \varphi(\omega_1, \omega_2, \omega_3)$, $\omega_1 = x_1$, $\omega_2 = x_2$, $\omega_3 = x_3$,

$$\varphi_{11} + \varphi_{22} + \varphi_{33} = 0.$$

5.2. $< \alpha P_0 + P_1 >$ $(\alpha \geq 0)$:

$$u = \varphi(\omega_1, \omega_2, \omega_3), \quad \omega_1 = x_0 - \alpha x_1, \quad \omega_2 = x_2, \omega_3 = x_3,$$

$$(1 - \alpha^2)\varphi_{11} - \varphi_{22} - \varphi_{33} + \lambda\varphi_1\sqrt{\alpha^2\varphi_1^2 + \varphi_2^2 + \varphi_3^2} = 0.$$

5.3. $< D + \alpha Z >$ $(\alpha \in R)$: $u = \alpha \ln x_0 + \varphi(\omega_1, \omega_2, \omega_3)$,

$$\omega_1 = \frac{x_1}{x_0}, \quad \omega_2 = \frac{x_2}{x_0}, \quad \omega_3 = \frac{x_3}{x_0},$$

$$(\omega_1^2 - 1)\varphi_{11} + (\omega_2^2 - 1)\varphi_{22} + (\omega_3^2 - 1)\varphi_{33} + 2\omega_1\varphi_1 + 2\omega_2\varphi_2 + 2\omega_3\varphi_3 - \alpha -$$

$$\lambda\sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2}(\omega_1\varphi_1 + \omega_2\varphi_2 + \omega_3\varphi_3 - \alpha) = 0.$$

5.4. $< Z + \alpha P_0 >$ $(\alpha = \pm 1)$: $u = \alpha \ln x_0 + \varphi(\omega_1, \omega_2, \omega_3)$,

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3,$$

$$\varphi_{11} + \varphi_{22} + \varphi_{33} - \alpha\lambda\sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2} = 0.$$

5.5. $< Z + \alpha P_0 + P_1 > (\alpha \in R): \quad u = x_1 + \varphi(\omega_1, \omega_2, \omega_3),$

$$\omega_1 = x_0 - \alpha x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3,$$

$$(1-\alpha^2)\varphi_{11} - \varphi_{22} - \varphi_{33} + \lambda\varphi_1\sqrt{\alpha^2\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - 2\alpha\varphi_1 + 1} = 0.$$

5.6. $< J_{12} + \alpha D + \beta Z >$, where $\alpha > 0$, $\beta \in R$ or $\alpha = 0$, $\beta \geq 0$:

$$u = \beta \arctan\frac{x_2}{x_1} + \varphi(\omega_1, \omega_2, \omega_3),$$

$$\omega_1 = \frac{x_1^2 + x_2^2}{x_0^2}, \quad \omega_2 = \frac{x_3}{x_0}, \quad \omega_3 = 2\alpha\arctan\frac{x_2}{x_1} - \ln(x_1^2 + x_2^2),$$

$$4\omega_1(\omega_1 - 1)\varphi_{11} + (\omega_2^2 - 1)\varphi_{22} - 4(1+\alpha^2)\omega_1^{-1}\varphi_{33} +$$

$$4\omega_1\omega_2\varphi_{12} + 8\varphi_{13} + (6\omega_1 - 4)\varphi_1 + 2\omega_2\varphi_2 - \lambda\varepsilon\times$$

$$\sqrt{4\omega_1\varphi_2^2 + \varphi_2^2 + 4(1+\alpha^2)\omega_1^{-1}\varphi_3^2 - 8\varphi_1\varphi_3 + 4\alpha\beta\omega_1^{-1}\varphi_3 + \beta^2\omega_1^{-1}}\times$$

$$(2\omega_1\varphi_1 + \omega_2\varphi_2) = 0,$$

where $\varepsilon =$sign x_0.

5.7. $< J_{12} + P_0 + \alpha Z > (\alpha \in R): \quad u = \alpha x_0 + \varphi(\omega_1, \omega_2, \omega_3),$

$$\omega_1 = x_0 + \arctan\frac{x_1}{x_)2}, \quad \omega_2 = x_1^2 + x_2^2, \quad \omega_3 = x_3,$$

$$(1-\omega_2^{-1})\varphi_{11} - 4\omega_2\varphi_{22} - \varphi_{33} - 4\varphi_2 + \lambda\sqrt{\omega_2^{-1}\varphi_1^2 + 4\omega_2\varphi_2^2 + \varphi_3^2}(\alpha + \varphi_1) = 0.$$

5.8. $< J_{12} + \alpha P_0 + P_3 + \beta Z >$, where $\alpha > 0$, $\beta \in R$ or $\alpha = 0$, $\beta \geq 0$:

$$u = \beta x_3 + \varphi(\omega_1, \omega_2, \omega_3), \quad \omega_1 = x_0 - \alpha x_3,$$

$$\omega_2 = x_1^2 + x_2^2, \quad \omega_3 = \arctan\frac{x_1}{x_2} + x_3,$$

$$(1-\alpha^2)\varphi_{11} - 4\omega_2\varphi_{22} - (1+\omega_2^{-1})\varphi_{33} - 4\varphi_2 + 2\alpha\varphi_{13} +$$

$$\lambda\varphi_1\sqrt{\alpha^2\varphi_1^2 + 4\omega_2\varphi_2^2 + (1+\omega_2^{-1})\varphi_3^2 - 2\alpha\varphi_1\varphi_3 + \beta^2 - 2\alpha\beta\varphi_1 + \beta^2 + 2\beta\varphi_3} = 0.$$

6 Multiplying the solutions

Let (c_{ij}) be an orthogonal matrix of order three and $d_0, d_j (j = 1, 2, 3)$ be arbitrary real numbers. If $u = f(x_0, x_1, x_2, x_3)$ is a solution of (1) then the function

$$u = \varepsilon f(Ay_0, Ay_1, Ay_2, Ay_3) + B, \tag{7}$$

(here $\varepsilon = \pm 1, y_0 = x_0 + d_0, y_i = \sum c_{ij}x_j + d_i; A, B$ are arbitrary real numbers and moreover $A > 0, (i, j = 1, 2, 3)$ is also the solution of this equation. All solutions of (1), obtained from the solution $u = f(x_0, x_1, x_2, x_3)$, as a result of application of transformations from the symmetry group (with the Lie algebra F) of this equation, are exhausted by functions of the form (6.1). Because the formula (6) is not concerned with the structure of the solution, it is impossible to state that all solutions of (1) obtained by means of (6) are different. For example, homogeneous transformations corresponding to matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

do not change the solution $u = \ln\{x_1^2 + x_2^2\}$. According to what has been said, it is also necessary to utilize formulae obtained for some sets of solutions in addition to (6).

If $u = f(x_0, x_1^2 + x_2^2 + x_3^2)$, then as a result of multiplying it, it is possible to receive only solutions of the following form:

$$u = \varepsilon f(Ay_0, A^2(y_1^2 + y_2^2 + y_3^2)) + B,$$

where $y_i = x_i + d_i, (i = 0, 1, 2, 3)$. All the solutions obtained by multiplying solution $u = f(x_0, x_1^2 + x_2^2, x_3)$ could be represented in the form

$$u = \varepsilon f(Ay_0, A^2(y_1^2 + y_2^2), Ay_3)) + B,$$

where $\varepsilon = \pm 1, A, B$ are arbitrary real numbers, and $A > 0,\ y_0 = x_0 + d_0$,

$$y_1^2 + y_2^2 = [x_1\cos\varphi - x_2\sin\varphi + d_1]^2 + [(x_1\sin\varphi + x_2\cos\varphi)\cos\psi - -x_3\sin\psi + d_2]^2,$$

$$y_3 = x_1\sin\varphi\sin\psi + x_2\sin\psi\cos\varphi + x_3\cos\psi + d_3.$$

Here d_0, d_1, d_2, d_3 are arbitrary real numbers, and the parameters φ, ψ accept arbitrary values in the interval $[0, 2\pi)$.

References

[1] Fushchych W.I. and Serova M.M., On some exact solutions of the nonlinear equations which are invariant under Euclid and Galilei groups in Algebraic-theoretical methods in mathematical physics problemsl, Institute of Mathematical, Kiev, 1983, 24–54.

[2] Fushchych W.I., Shtelen V.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer Academic Publishers, 1993, 400 p.

[3] Ovsyannikov L.V., Group Analysis of Differential Equations, Academic Press, New York, 1982, 400 p.

[4] Patera J., Winternitz P. and Zassenhaus H., Continuous subgroups of the fundamental groups of physics, I. General method and the Poincarè qroup, *J. Math. Phys.*, 1975, V.16, N 8, 1597–1624.

[5] Fushchych W.I., Barannyk L.F. and Barannyk A.F., Subgroup Analysis of the Galilei, Poincarè Groups and Reduction of Nonlinear Equation, Naukova Dumka, Kiev, 1991, 304 p.

[6] Lang S., Algebra, Addison-Wesly publishing company reading, New York, 1965.

[7] Olver P., Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986, 400 p.