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Non–Lie Ansatzes for Nonlinear Heat Equations

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Abstract

Operators of non–local symmetry are used to construct exact solutions of nonlinear heat equations.

A method for finding of new classes of ansatzes reducing nonlinear wave equations to a systems of ordinary differential equations was suggested in [1]. This approach is based on non–local symmetry of differential equations. In the present paper we apply this method to the nonlinear heat equation.

Let us consider the equation

$$u_t - u_{xx} = H(u), \tag{1}$$

where $H(u)$ is some smooth function.

The following system

$$\begin{aligned} &v^1_t + v^1_3 v^2 = v^2_3 v^1, \\ &v^2 - v^1_3 v^1 = H(x_3), \end{aligned} \tag{2}$$

where $u \equiv x_3$, $\partial u/\partial x \equiv v^1$, $\partial u/\partial t \equiv v^2$, corresponds to the equation (1) if we use the approach suggested in [1].

Theorem 1 *The system (2) is Q–conditionally invariant with respect to the operator*

$$\begin{aligned} Q = \partial_{x_3} + 2F\exp\left(-F^2\right)v^1\partial_{v^1} + \\ \left(2F\exp\left(-F^2\right)v^2 + \frac{\exp\left(-F^2\right)v^2 - 1}{F}\right)\partial_{v^2} \end{aligned} \tag{3}$$

if

$$H(x_3) = \exp\left(F^2(x_3)\right),$$

where $F(x_3) = \Phi^{-1}(x_3)$, $\Phi(x_3) = \int \exp\left((x_3)^2\right) dx_3$.

Proof. We use the criterion of Q–conditional invariance [2, 3]. Thus we have

$$\begin{aligned}&\widetilde{Q}(v^2 - v^1_3 v^1 - \exp\left(F^2(x_3)\right)) = \\ &\qquad 2F\exp\left(-F^2\right)v^2 + \left(\exp\left(-F^2\right)v^2 - 1\right)/F - \\ &\qquad v^1\left(2F\exp\left(-F^2\right)v^1 - 4F^2\exp\left(-2F^2\right)v^1 + 2F\exp\left(-F^2\right)v^1_3\right) - \\ &\qquad 2v^1_3 F\exp\left(-F^2\right)v^1 - 2F,\end{aligned} \tag{4}$$

where $\widetilde{Q}$ is the prolongation of the operator Q.

Taking into account

$$v^2 = 2F\exp\left(-F^2\right)\left(v^1\right)^2 + \exp\left(F^2\right),$$

$$v^1_3 = 2F\exp\left(-F^2\right)v^1,$$

$$v^2_3 = 2F\exp\left(-F^2\right)v^2 + \left(\exp\left(-F^2\right)v^2 - 1\right)/F$$

we obtain

$$\begin{aligned}&\widetilde{Q}(v^2 - v^1_3 v^1 - \exp\left(F^2(x_3)\right)) = \\ &\qquad 2F\exp\left(-F^2\right)\Big(2F\exp\left(-F^2\right)\left(v^1\right)^2 + \exp\left(F^2\right)\Big) + \\ &\qquad \Big(\exp\left(-F^2\right)\Big(2F\exp\left(-F^2\right)\left(v^1\right)^2 + \exp\left(F^2\right)\Big) - 1\Big)/F - \\ &\qquad v^1\left(2\exp\left(-2F^2\right)v^1 - 4F^2\exp\left(-F^2\right)v^1 + 4F^2\exp\left(-2F^2\right)v^1\right) - \\ &\qquad 4F^2\exp\left(-2F^2\right)\left(v^1\right)^2 - 2F \equiv 0.\end{aligned}$$

Similarly we receive

$$\widetilde{Q}\left(v^1_t + v^1_3 v^2 - v^2_3 v^1\right) \equiv 0. \tag{5}$$

Q.E.D.

The operator (3) generates the ansatz

$$\begin{aligned}v^1 &= \exp\left(F^2\right)\varphi_1(t),\\ v^2 &= \exp\left(F^2\right)(2F\varphi_2(t) + 1),\end{aligned} \tag{6}$$

where φ_1, φ_2 are unknown functions.

Substitution of (6) into (2) yields the system of two ordinary differential equations for φ_1, φ_2

$$\begin{aligned}d\varphi_1/dt &= 2\varphi_1\varphi_2,\\ \varphi_2 &= \varphi_1^2\end{aligned} \tag{7}$$

whose general solution has the form

$$\varphi_1 = 1/\sqrt{C-4t},$$
$$\varphi_2 = 1/(C-4t). \tag{8}$$

Integrating the overdetermined but compatible system

$$u_x = \exp\left(F^2(u)\right)/\sqrt{C-4t},$$
$$u_t = \exp\left(F^2\right)\left(2F/(C-4t)+1\right) \tag{9}$$

we get the exact solution of the nonlinear heat equation with the function $H(u) = \exp\left(F^2(u)\right)$

$$u = \Phi\left(\frac{\pm 6x - \left(\sqrt{C-4t}\right)^3 + C_1}{6\sqrt{C-4t}}\right). \tag{10}$$

where $\Phi(z) = \int \exp z^2 dz$. The maximal invariance algebra of the equation

$$u_t - u_{xx} = \exp\left(F^2(u)\right) \tag{11}$$

is a 2–dimensional Lie algebra whose basic elements are given by the formulae

$$P_x = \partial_x, \qquad P_t = \partial_t.$$

It is obvious that the solution (10) is not an invariant solution.

Next we consider the equation

$$u_{xx} = F(u_t). \tag{12}$$

The associated system has the form

$$v^1_2 = v^2_1,$$
$$v^2_2 = F(v^1), \tag{13}$$

where $v^1 \equiv u_t$, $v^2 \equiv u_x$, $x_1 \equiv t$, $x_2 \equiv x$. The following statement has been proved by means of Lie‘s algorithm [2].

Theorem 2 *The system (13) is invariant with respect to the operator*

$$Q = -x_1\partial x_1 + v^2\partial x_2 + v^1\partial v_1 \tag{14}$$

if

$$F(v^1) = 1/\ln v^1.$$

The ansatz corresponding to the operator (14) is as follows

$$v^1 = \varphi_1(v^2)/x_1,$$
$$v^2 = x_2/(\varphi_2(v^2) - \ln x_1). \tag{15}$$

Substituting (15) into (13) we obtain the system of ordinary differential equations

$$\begin{aligned} &\ln\varphi_1 - \varphi_2 = v^2 d\varphi_2/dv^2, \\ &d\varphi_1/dv^2 = v^2. \end{aligned} \tag{16}$$

The general solution of the system (16) has the form

$$\begin{aligned} &\varphi_1 = \frac{1}{2}\left(\left(v^2\right)^2 + C\right), \\ &\varphi_2 = \ln\frac{\left(\left(v^2\right)^2 + C\right)}{2e^2} + \frac{C_1}{v^2} + \frac{2C}{v^2}\int\frac{dv^2}{(v^2)^2 + C}. \end{aligned} \tag{17}$$

Setting $C = 0$ in (17) we obtain the system

$$\begin{aligned} &u_t = \left(u_x\right)^2/(2t), \\ &u_x = 2x/\left(\ln\frac{\left(u_x\right)^2}{2e^2} + \frac{C_1}{u_x} - \ln t\right). \end{aligned} \tag{18}$$

To construct the solution of the equation

$$u_{xx} = 1/\ln u_t \tag{19}$$

it is necessary to integrate the system (18).

The following formula

$$\begin{aligned} &\exp\left(1/z\right) = \theta^2/(2t), \\ &\theta = 2x/\left(\ln\frac{\theta^2}{2e^2} + \frac{C_1}{\theta} - \ln t\right) \end{aligned} \tag{20}$$

gives a parametric solution of the equation

$$z_t + \left(z^{-2}\exp\left(z^{-1}\right)z_x\right)_x = 0$$

where θ is a parameter.

It should be noted that the ansatzes (6) and (15) which reduce the equation (11) and (19) respectively cannot be obtained by means of the classical Lie method.

References

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