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Neumann-Bogolyubov-Rosochatius Oscillatory Dynamical Systems and their Integrability Via Dual Moment Maps. Part I

*A. PRYKARPATSKY**, *O. HENTOSH**, *M. KOPYCH*** and *R. SAMULIAK***

**Institute for Applied Problems of Mechanics and Mathematics of the National Academy of Sciences of Ukraine, 3b Naukova Str., UA–290060 Lviv, Ukraina*

***State University "L'viv Polytechnic", Dept. of Advanced Mathematics, 12 St. Bandera Str., UA–290013 Lviv, Ukraina*

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Abstract

The finite-dimensional invariant subspaces of the solutions of intergrable by Lax infinite-dimensional Benney-Kaup dynamical system are presented. These invariant subspaces carry the canonical symplectic structure, with relation to which the Neumann type dynamical systems are Hamiltonian and Liouville intergrable ones. For the Neumann-Bogolyubov and Neumann-Rosochatius dynamical systems, the Lax-type representations via the dual moment maps into some deformed loop algebras as well as the finite hierarchies of conservation laws are constructed.

1 Introduction

The Neumann-type dynamical systems consist of harmonic oscillators with some external forcing, constrained to move on a unit sphere in the configuration space. This paper is concerned with finite-dimensional invariant subspaces of solutions to some infinite-dimensional Lax-type integrable dynamical system called the Benney-Kaup one. The finite-dimensional invariant subspace carries the canonical symplectic structure, with relation to which the Neumann-type dynamical systems are Hamiltonian and Liouville integrable ones.

The principle purpose of the present work is to provide a systematic procedure for the Neumann-type dynamical systems to be treated basing on flows in loop algebras, the Novikov-Lax reduction approach and the use of moment maps. The latter was recently systematically developed in [1, 2], the Novikov-Lax reduction was devised in [3, 4] and further thoroughly augmented in [5] for the case of nonlocal Lagrangian submanifolds, generated by the spectrum of an associated Lax-type operator. As a finite result, we get a possibility to present a Lax-type representation for the Neumann-type dynamical systems including the important Neumann-Bogolyubov-Rosochatius system [6] in the form firstly found by J.Moser [7] but including the spectral parameter lying on the circle $\mathbf{S}^1$.

2 The Lie-algebraic setting of the Benney-Kaup dynamical systems

Within the Lie-algebraic approach [11], a large class of integrable systems on an infinite-dimensional functional manifold M can be derived by constructing a moment map into the dual space of some loop algebras. Let $\tilde{\mathcal{G}}$ denote the semi-infinite formal loop Lie algebra over a semisimple algebra $\mathcal{G}$, i.e., an element $a(\lambda) \in \tilde{\mathcal{G}}$, if

$$a(\lambda) = \sum_{j<<\infty} u_j \lambda^j, \quad u_j \in \mathcal{G} \tag{1}$$

for all $j << \infty$, where $\lambda \in \mathbf{C}$ is a parameter. We use the vector space direct sum splitting

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_- \tag{2}$$

where $\tilde{\mathcal{G}}_+$ denotes the Lie subalgebra of polynomials in $\lambda \in \mathbf{C}, \tilde{\mathcal{G}}_-$ is the Lie subalgebra of strictly negative formal series in $\tilde{\mathcal{G}}$. Under the pairing

$$\langle u(\lambda), l(\lambda)\rangle_p = res_{\lambda=0}\, \lambda^p Sp(u(\lambda)l(\lambda)), \quad u, l \in \tilde{\mathcal{G}},\ p \in \mathbf{Z}, \tag{3}$$

we can build dual spaces $\tilde{\mathcal{G}}^*_{+,p}$ and $\tilde{\mathcal{G}}^*_{-,p} \subset \tilde{\mathcal{G}}^*$, where

$$\tilde{\mathcal{G}}^*_{+,p} \simeq \tilde{\mathcal{G}}_- \lambda^{-p}, \quad \tilde{\mathcal{G}}^*_{-,p} \simeq \tilde{\mathcal{G}}_+ \lambda^{-p}. \tag{4}$$

Hence, over the loop algebra $\tilde{\mathcal{G}}$ there exists the canonical $\mathcal{R}$-structure giving rise to a natural Poisson structure given by the Lie-Poisson structure of $\tilde{\mathcal{G}}^*$. To augment this construction further, we need to involve into our analysis the standard central extension of the loop algebra $\tilde{\mathcal{G}}$ via the two-cocycle $\omega_p(a,b) = (a, db/dx)_p$, $a, b \in \tilde{\mathcal{G}}$, $p \in \mathbf{Z}$, where we will have some mapping $x \in \mathbf{R}/2\pi\mathbf{Z} \to \tilde{\mathcal{G}}$, transforming the loop algebra $\tilde{\mathcal{G}}$ into the current Lie algebra $\hat{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbf{C}$ on the circle $\mathbf{S}^1$. The current Lie algebra $\hat{\mathcal{G}}$ is a metrized Lie algebra of currents on the circle $\mathbf{S}^1$ with a nondegenerate scalar product

$$(a,b)_p := res_{\lambda=0}\lambda^p \int_0^{2\pi} dx\, Sp(a(\lambda)b(\lambda)), \tag{5}$$

in relation to which we have $\hat{\mathcal{G}}^* \simeq \hat{\mathcal{G}}$ together with the Lie subalgebra direct sum splitting:

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-. \tag{6}$$

We can also convince ourselves that the analog of (4) takes place:

$$\hat{\mathcal{G}}^*_{+,p} \simeq \hat{\mathcal{G}}_- \lambda^{-p}, \quad \hat{\mathcal{G}}^*_{-,p} \simeq \hat{\mathcal{G}}_+ \lambda^{-p} \tag{7}$$

for all $p \in \mathbf{Z}$.

To find the natural hierarchy of Lie-Poisson structures on the current Lie algebra $\hat{\mathcal{G}}$ on $\hat{\mathcal{G}}$, we compute the following hierarchies of $\mathcal{R}$-structures and brackets on the $\mathcal{D}(\hat{\mathcal{G}}^*)$ for all $p \in \mathbf{Z}$:

$$[a,b]_{\mathcal{R}} := [\mathcal{R}a, b] + [a, \mathcal{R}b], \quad a, b \in \hat{\mathcal{G}},$$

$$\mathcal{R} = (P_+ - P_-), \quad P_\pm \hat{\mathcal{G}} := \hat{\mathcal{G}}_\pm, \quad P_\pm^2 = P\pm;$$

$$\{\gamma, \mu\}_{\theta_p}(l) := (l_p, [\nabla\gamma(l_p), \nabla\mu(l_p)_{\mathcal{R}})_0 = \tag{8}$$

$$(l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}})_p = (\nabla\gamma(l), \theta_p \nabla\mu(l))_0,$$

where by definition $l_p := \lambda^p l \in \hat{\mathcal{G}}^*$, if the brackets above are nontrivial.

Let the phase space manifold $M \subset C^\infty(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{R}^m)$ be defined by means of the moment map $M \ni u :\to l := l[u, \lambda] \in \hat{\mathcal{G}}$, with the fixed set of Casimir gauge type invariants

$$I(\hat{\mathcal{G}}) := \{\gamma \in \mathcal{D}(\hat{\mathcal{G}}) : [l - d/dx, \nabla\gamma(l)] = 0,\ x \in \mathbf{R}/2\pi\mathbf{Z}\} \tag{9}$$

in relation to the standard scalar product $(.,.)_0$ upon the current Lie algebra $\hat{\mathcal{G}}$. Since the Lie-Poisson brackets $\{.,.\}_{\theta_p}$, $p \in \mathbf{Z}$, are generally not invariant on the manifold M, we must apply the well-known Dirac procedure [11, 12] to the bracket (8) extended on some manifold M. The result is found to be

$$\{\gamma, \mu\}_{\theta_{p[u]}} := \{\gamma, \mu\}_{\theta_p} - \sum_{j,k=1}^{n_p} \{\gamma, \Phi_j^{(p)}\}_{\theta_p} \|\{\Phi^{(p)}, \Phi^{(p)}\}\|_{(j,k)}^{-1} \{\Phi_k^{(p)}, \mu\}_{\theta_p}, \tag{10}$$

where $M := \{u \in M_{ext} : \Phi_j^{(p)} \equiv 0,\ j = \overline{1, n_p}\}$. It is important to observe here that the shift $l \to l_p := \lambda^p l \in \hat{\mathcal{G}}$, $p \in \mathbf{Z}$, in the second bracket of (8) is in full agreement with the Poisson invariance of the manifold M, determined by the set of Casimir invariants $I(\hat{\mathcal{G}})$ for all $p \in \mathbf{Z}$, that is $I(\hat{\mathcal{G}}; (.,.)_p) \equiv I(\hat{\mathcal{G}})$, where

$$I(\hat{\mathcal{G}}; (.,.)_p) := \{\gamma \in \mathcal{D}(\hat{\mathcal{G}}) : [l_p - \lambda^p d/dx, \nabla\gamma(l_p)] = 0,\ x \in \mathbf{R}/2\pi\mathbf{Z}\}. \tag{11}$$

The scalar Benney-Kaup hierarchy of nonlinear dynamical systems on the functional manifold $M \subset C^\infty(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{R}^2)$ is associated [12] with the element $l \in \hat{\mathcal{G}}^*$ if $\mathcal{G} := sl(2; \mathbf{R})$ and

$$l(x; \lambda) := \sigma^1(u + \lambda v - \lambda^2) + \sigma^2,\ (u, v)^\tau \in M. \tag{12}$$

This element admits the extension $l \to l_p = \lambda^p l$ for $p = -2$ with the constraint functions having the form:

$$\Phi_1 = u_{2,1}^{(-2)} - 1, \quad \Phi_2 = u_{3,1}^{(-2)}, \quad \Phi_3 = u_{2,0}^{(-2)}, \quad \Phi_4 = u_{3,0}^{(-2)},$$

$$\Phi_5 = u_{1,-1}^{(-2)} + 1, \quad \Phi_6 = u_{2,-1}^{(-2)}, \quad \Phi_7 = u_{3,-1}^{(-2)}, \tag{13}$$

Here, by definition, for all $p \in \mathbf{Z}$

$$l_p(x; \lambda) := \sum_{j,a} u_{a,k-p}^{(p)}(x) \sigma^a \lambda^{p-k-1}, \tag{14}$$

and

$$\mathcal{G}^* := span_{\mathbf{R}}\{\sigma^a : a = \overline{1, dim\,\mathcal{G}}\} \tag{15}$$

is the standard dual space to the semisimple Lie algebra $\mathcal{G} := span_{\mathbf{R}}\{\sigma^a : a = \overline{1, dim\,\mathcal{G}}\}$, the element (12) being belonged to $\hat{sl}(2;\mathbf{R})$, where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

and functions $u_{a,k}^{(0)} = u_{a,k-p}^{(p)} \in M_{ext}$, $a = \overline{1,3}$, $k \in \mathbf{Z}$. The phase space manifold $M \subset M_{ext}$ is defined as follows:

$$M = \{(u,v)^\tau \in M_{ext} : \Phi_j = 0,\ j = \overline{1,7}, u_{1,1}^{(-2)} = u,\ u_{1,0}^{(-2)} = v\}. \tag{17}$$

The constraint matrix $\hat{\Phi} := \|\{\Phi_i, \Phi_j\}_{-2}\|$, $i,j = \overline{1,4}$, is nondegenerate, i.e., the constraints $\{\Phi_j : j = \overline{1,4}\}$ are of the second class, and the embedding (17) of the basic manifold M into the extended manifold M_{ext} is coisotropic [13]. As a result, we can obtain the Poisson structure on the manifold M, being given by the following implectic operator upon $T^*(M)$:

$$\theta_{-2}[u,v] = \begin{pmatrix} 0 & \partial^3/2 - (\partial u + u\partial) \\ \partial^3/2 - (\partial u + u\partial) & (\partial v + v\partial) \end{pmatrix}. \tag{18}$$

In analogous way we can compute the Poisson structure $\theta_{-3}[u,v]$ on the manifold M:

$$\begin{aligned}
\{u,u\}_{\theta_{-3}[u,v]} &= -\frac{1}{8}(-\partial^3 + 2u\partial + 2\partial u)\partial^{-1}(-\partial^3 + 2u\partial + 2\partial u), \\
\{u,v\}_{\theta_{-3}[u,v]} &= \frac{1}{4}(-\partial^3 + 2u\partial + 2\partial u)\partial^{-1}(v\partial + \partial v), \\
\{v,v\}_{\theta_{-3}[u,v]} &= \frac{1}{2}(\partial v\partial^{-1}v\partial - \partial v^2 - v^2\partial + v\partial v + \partial^3) - (\partial u + u\partial).
\end{aligned} \tag{19}$$

If the following implectic operators on M are defined by

$$\theta = \begin{pmatrix} -(2\partial v + 2v\partial) & 4\partial \\ 4\partial & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} -\partial^3 + 2(\partial u + u\partial) & 0 \\ 0 & 4\partial \end{pmatrix}, \tag{20}$$

then we can obtain easily $\theta_{-2}[u,v] = (1/2)\theta^{(2)}$, $\theta_{-3}[u,v] = (1/2)\theta^{(3)}$, where $\theta^{(n)} := \theta\Lambda^n$, $\Lambda = \theta^{-1}\eta$, $n \in \mathbf{Z}$. Since the implectic operators $\theta_{-2}[u,v]$ and $\theta_{-3}[u,v]$ are compatible by construction, i.e., for $\lambda \in \mathbf{R}$ the operator $\theta_{-2}[u,v] + \lambda\theta_{-3}[u,v]$ is also implectic on the manifold M, it follows that the implectic (θ,η)- pair (20) is also compatible. The associated hierarchy of Lax integrable nonlinear Benney-Kaup dynamical systems is obtained easily from the following generating formula:

$$du/dt_n = -(\Lambda^*)^n(u_x, v_x) \tag{21}$$

for all $n \in \mathbf{Z}$. As a result, for $n = 1$ we obtain the Benney-Kaup nonlinear hydrodynamic system of equations:

$$\begin{aligned}
du/dt_1 &= v_{xxx} - 4uv_x - 2u_x v, \\
dv/dt_1 &= -4u_x - 6vv_x.
\end{aligned} \tag{22}$$

Since $\Lambda^{*n}(u_x, v_x)^{\tau} = \theta\, grad\, \gamma_n = \eta\, grad\, \gamma_{n-1}$ for $n \in \mathbf{Z}$, where $\gamma_n \in \mathcal{D}(M)$, $n \in \mathbf{Z}$, is an infinite involutive hierarchy of conservation laws of the hydrodynamic system (22), we state its complete integrability on each invariant finite-dimensional submanifold $M_N \subset M$ generated as follows:

$$M'_N := \{u \in M : grad\mathcal{L}'_N[u, v] = 0\}. \tag{23}$$

Here the Lagrangian $\mathcal{L}'_N := -\gamma_{N(\gamma)+1} + \sum\limits_{j=0}^{N(\gamma)} a_j\gamma_j + \sum\limits_{j=0}^{N(\lambda)} b_j\lambda_j$, where $\lambda_j \in \mathcal{D}(M)$, $j = \overline{0, N(\lambda)}$, are the nondegenerated eigenvalues of the generalized periodic spectral problem $df/dx + l[u, v; \lambda]f = 0$, $f \in L_{\infty}(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{C}^2)$ on the real axis $\mathbf{R}$; $a_j \in \mathbf{R}$, $j = \overline{1, N(\gamma)}$, $b_j \in \mathbf{R}$, $k = \overline{1, N(\lambda)}$, are some arbitrary numbers, $\mathbf{Z}_+ \ni N(\gamma) < \infty$, $\mathbf{Z}_+ \ni N(\lambda) < \infty$.

For further convenience, let us list some conservation laws in exact form:

$$\gamma_0 = \frac{1}{2}\int\limits_0^{2\pi} dx\, v, \quad \gamma_1 = \frac{1}{2}\int\limits_0^{2\pi} dx\,(2u + v^2/2),$$

$$\gamma_2 = \frac{1}{2}\int\limits_0^{2\pi} dx\,(uv + v^3/4), \tag{24}$$

$$\gamma_2 = \frac{1}{2}\int\limits_0^{2\pi} dx\,[(u + v^2/4)^2 + v_x^2/4 + v(u + v^2/4)], \ldots .$$

Note. In general, we just notice that the Casimir functional hierarchy satisfied the main equation from (11) in the dual space $\hat{\mathcal{G}}^*$:

$$\frac{d}{dx} grad\, \gamma(l) = [l, grad\, \gamma(l)] \tag{25}$$

for $l \in \hat{\mathcal{G}}^*$, $\gamma \in I(\hat{\mathcal{G}}^*)$ by definition. Since we know that the functional $\gamma(l) := Sp\, S(x; \lambda)$ is the Casimir one, where the matrix $S(x; \lambda)$ is monodromy matrix of the above mentioned generalized spectral problem $df/dx - l[u; \lambda]f = 0$, $f \in L_{\infty}(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{C}^2)$, we can find the following important Novikov-Lax equation on the monodromy matrix $S(x; \lambda)$:

$$dS/dx = [l, S], \tag{26}$$

as $grad\, \gamma(l) \equiv S(x; \lambda)$, $x \in \mathbf{R}$, $\lambda \in \mathbf{C}$. This equation has an infinite hierarchy of solutions in general , polynomial in $\lambda \in \mathbf{C}$, which can be found by means of some reccurent procedure. Due to the following important relationship on the manifold M:

$$grad\, \gamma(l)[u, v] = (s_{12}(x; \lambda), \lambda s_{12}(x; \lambda))^{\tau} \tag{27}$$

we have an easy possibility to find all the infinite hierarchy of conservation laws (24) through simple expansions of (27) in degrees of the spectral parameter $\lambda \in \mathbf{C}$. We will not stop upon this problem in the article more.

3 The Novikov-Lax finite-dimensional invariant reductions on nonlocal submanifolds

The submanifold $M'_N \subset M$ defined in (23) is nonlocal in general due to the presence of the eigenvalues $\lambda_j \in \mathcal{D}(M)$, $j = \overline{0, N(\lambda)}$, as smooth but nonlocal functionals on M. To cap with this difficulty, we suggest firstly, as in [5, 14], to extend the phase space M to $M \times W^{2N(\lambda)+2}$, where the space W is some Sobolev subspace in the space $L_\infty(\mathbf{R}/2\pi\mathbf{Z};\mathbf{C}^2)$ of eigenfunctions with the above fixed eigenvalues $\lambda_j \in \mathcal{D}(M)$, $j = \overline{0, N(\lambda)}$, and, secondly, to introduce into the system of invariants $\{\gamma_j \in \mathcal{D}(M),\ j = \overline{0, N(\gamma)};\ \lambda_k \in \mathcal{D}(M) : k = \overline{0, N(\lambda)}\}$ an additional set of norming functionals as follows:

$$\mathcal{L}'_N \to \mathcal{L}_N : -\gamma_{N(\gamma)+1} + \sum_{j=0}^{N(\gamma)} a_j\gamma_j + \sum_{j=0}^{N(\lambda)} b_j\lambda_j + \sum_{j=0}^{N(\lambda)} c_j s_j. \tag{28}$$

Here

$$s_j = (f_j^*, \partial l/\partial\lambda f_j) = \int_0^{2\pi} dx\, f_{j,2}^*(v + 2\lambda_j) f_{j,1} \in \mathcal{D}(M \times W^{2N(\lambda)+2}), \tag{29}$$

$c_j \in \mathbf{R}$, $j = \overline{0, N(\lambda)}$, are arbitrary numbers, and

$$\begin{aligned} \frac{d}{dx} f_j \quad &- \quad l[u, v; \lambda_j] f_j = 0, \quad f_j = (f_{j,1}, f_{j,2})^\tau, \\ \frac{d}{dx} f_j^* \quad &+ \quad l^*[u, v; \lambda_j] f_j^* = 0, \quad f_j^* = (f_{j,1}^*, f_{j,2}^*)^\tau, \end{aligned} \tag{30}$$

where $f_j, f_j^* \in W$, $j = \overline{0, N(\lambda)}$, are the eigenfunctions of the right and adjoint eigenvalue problems for the generalized operator $d/dx + l[u, v; \lambda]$, acting in $L_\infty(\mathbf{R}/2\pi\mathbf{Z};\mathbf{C}^2)$.

As a result, we can construct the following local finite-dimensional submanifold $M_N(W) \subset M(W) := M \times W^{2N(\lambda)+2}$:

$$M_N(W) = \{(u, v; f, f^*)^\tau \in M(\times W) : grad\,\mathcal{L}_N[u, f, f^*] = 0\}, \tag{31}$$

where, by definition, $f := (f_0, f_2, \ldots, f_{N(\lambda)})$, $f^* := (f_0^*, f_2^*, \ldots, f_{N(\lambda)}^*) \in W^{N(\lambda)+1}$. Here we need to note that the gradient-operation in (31) is taken in relation to the suitably indicated argument variables, that is, to $(u, f, f^*)\tau \in M(W)$. Let us consider the following Lagrangian:

$$\overline{\mathcal{L}_N} = -\frac{1}{2}\gamma_1 + a_0\gamma_0 + \sum_{j=0}^{N(\lambda)} b_j\lambda_j + \sum_{j=0}^{N(\lambda)} c_j s_j, \tag{32}$$

and the condition $b_j = s_j$, $j = \overline{0, N(\lambda)}$, be satisfied upon the submanifold $\overline{M}_N(W)$:

$$\overline{M}_N(W) := \{(u, v; f, f^*) \in M(W) :$$

$$1 = \sum_{j=0}^{N(\lambda)} f_{j,2}^* f_{j,1}, \quad v/2 = a_0/2 + \sum_{j=0}^{N(\lambda)} (\lambda_j f_{j,2}^* f_{j,1}) + c_j f_{j,2}^* f_{j,1}, \tag{33}$$

$$c_j(\partial l^*/\partial\lambda)f_j^* = 0, \quad c_j(\partial l/\partial\lambda)f_j = 0\}.$$

The phase space variables $(f_j, f_j^*)^\tau \in W^2$, $j = \overline{1, N(\lambda)}$, satisfy the following eigenfunction equations:

$$\begin{aligned} df_j/d\tau &= l[u, v; \lambda_j]f_j, \\ df_j^*/d\tau &= -l^*[u, v; \lambda_j]f_j^*, \end{aligned} \tag{34}$$

where we have changed an independent variable: $\mathbf{R} \ni x \longleftrightarrow \tau \in \mathbf{R}$ for some further convenience. Let us put further $c_j \equiv 0, j = \overline{0, N(\lambda)}$, not constraining our investigation of any effective cases. Therefore, we can obtain in the case $\lambda_j^* = \lambda_j = \omega_j \in \mathbf{R}$, $j = \overline{0, N(\lambda)}$, $f_{j,2}^* = f_{j,1} = q_j \in W$, and

$$d^2q_j/d\tau^2 + \omega_j^2 q_j = q_j\omega_j v + uq_j \tag{35}$$

for all $j = \overline{0, N(\lambda)}$, under the prolonged naturally constraints (33):

$$\sum_{j=0}^{N(\lambda)} q_j^2 = 1, \quad v = 2\sum_{j=0}^{N(\lambda)} \omega_j q_j^2 + a_0, \quad \sum_{j=0}^{N(\lambda)} q_j(dq_j/d\tau) = 0. \tag{36}$$

To find an expression for the phase space variable $u \in M$, we must multiply each equation (35) by the adjoint element $q_j \in W$, $j = \overline{0, N(\lambda)}$, and sum up the result through $j = \overline{0, N(\lambda)}$:

$$\sum_{j=0}^{N(\lambda)} q_j(d^2q_j/d\tau^2) + \sum_{j=0}^{N(\lambda)} \omega_j^2 q_j^2 - 2\left(\sum_{j=0}^{N(\lambda)} \omega_j q_j^2\right)^2 = u. \tag{37}$$

Taking into account the last equation in (36) and differentiating it with respect to $\tau \in \mathbf{R}$, we obtain

$$\sum_{j=0}^{N(\lambda)} q_j(d^2q_j/d\tau^2) = -\sum_{j=0}^{N(\lambda)} (dq_j/d\tau)^2. \tag{38}$$

Whence and from (37) we get

$$u = -\sum_{j=0}^{N(\lambda)} (dq_j/d\tau)^2 + \sum_{j=0}^{N(\lambda)} \omega_j^2 q_j^2 - 2\left(\sum_{j=0}^{N(\lambda)} \omega_j q_j^2\right)^2. \tag{39}$$

As a result, we have got the Liouville integrable nonlinear Neumann-type oscillatory dynamical system

$$dq_j/d\tau = p_j, \quad dp_j/d\tau = -\omega_j^2 q_j + 2\omega_j \sum_{j=0}^{N(\lambda)} \omega_j q_j^2, \quad j = \overline{0, N(\lambda)}, \tag{40}$$

constrained to live on the cotangent space $T^*(\mathbf{S}^{N(\lambda)})$ to the sphere $\mathbf{S}^{N(\lambda)} = \{q \in \mathbf{R}^{N(\lambda)+1} : \sum_{j=0}^{N(\lambda)} q_j^2 = 1\}$, that is a simple exercise to the reader using the standard Dirac reduction procedure on a coisotropic submanifold defined by constraints of the second class [15, 16]. This means that the dynamical system (35) on $T^*(\mathbf{S}^{N(\lambda)})$ in the form

$$dq_j/d\tau = p_j, \quad dq_j/d\tau = -\omega_j^2 q_j + 2\omega_j q_j v + uq_j, \tag{41}$$

where $j = \overline{0, N(\lambda)}$, is the Hamiltonian system on the cotangent space $T^*(\mathbf{S}^{N(\lambda)})$ with respect to the canonical symplectic structure $\Omega^{(2)} := \sum_{j=0}^{N(\lambda)} dp_j \times dq_j$, being gotten from the oscillatory dynamical system (40) constrained to live on the cotangent space $T^*(\mathbf{S}^{N(\lambda)})$ to the sphere $\mathbf{S}^{N(\lambda)}$ with respect to the symplectic structure $\Omega^{(2)}$ obtained via the above mentioned Dirac procedure. Thereby we can write down the following Hamiltonian form of (41):

$$dq_j/d\tau = \{H, q_j\}_{\mathbf{S}^{N(\lambda)}}, \quad dp_j/d\tau = \{H, p_j\}_{\mathbf{S}^{N(\lambda)}}, \tag{42}$$

where $(q,p)^\tau \in T^*(\mathbf{S}^{N(\lambda)})$, $\{.,.\}_{\mathbf{S}^{N(\lambda)}}$ is the canonical Poisson bracket reduced on $T^*(\mathbf{S}^{N(\lambda)})$, and Hamiltonian

$$H = \frac{1}{2}\sum_{j=0}^{N(\lambda)} p_j^2 + \frac{1}{2}\sum_{j=0}^{N(\lambda)} \omega_j^2 q_j^2 - \frac{1}{8}v^2. \tag{43}$$

4 The Moser map and associated with it dual moment maps into loop Lie algebras

Let us consider the Novikov-Lax monodromy matrix equation (26) reduced upon an invariant finite-dimensional submanifold $M_N(W)$ built above. This reduction is called as the Moser map:$\mathcal{N} : S(\tau;\lambda) \to S(q,p;\lambda)$, where $(q,p)^\tau \in T^*(\mathbf{S}^{N(\lambda)})$, the monodromy matrix $S(\tau;\lambda)$ being given a priori traceless because of the τ-invariance of the manifold M. Accomplishing the Moser map of the monodromy matrix $S(\tau;\lambda)$, $\lambda \in \mathbf{C}$, on the finite-dimensional submanifold $M_N(W)$ built above, we can state the following

Theorem 1. *The monodromy matrix $S(q,p;\lambda)$ reduced on the invariant submanifold $M_N(W)$ is given in the componentwise form as follows [12]:*

$$\begin{aligned}
s_{12}(q,p;\lambda) &= \sum_{j=0}^{N(\lambda)} \frac{q_j^2}{\lambda - \omega_j}, \\
s_{21}(q,p;\lambda) &= -\frac{1}{2}\frac{d^2 S_{12}(q,p;\lambda)}{d\tau^2} + s_{12}(q,p;\lambda)[\lambda v(q,p) + u(q,p) - \lambda^2] = \\
&\quad -\sum_{j=0}^{N(\lambda)} \frac{p_j^2}{\lambda - \omega_j} + \sum_{j=0}^{N(\lambda)} \omega_j q_j^2 - \lambda + a_0, \\
s(q,p;\lambda) &= -\frac{1}{2}\frac{dS_{12}(q,p;\lambda)}{d\tau} = -\sum_{j=0}^{N(\lambda)} \frac{q_j p_j}{\lambda - \omega_j},
\end{aligned} \tag{44}$$

where, by definition, the matrix

$$S(q,p;\lambda) := \begin{pmatrix} s(q,p;\lambda) & s_{12}(q,p;\lambda) \\ s_{21}(q,p;\lambda) & -s(q,p;\lambda) \end{pmatrix} \tag{45}$$

is considered to be normalized by means of multiplying by some invariant functional on $M(W)$.

$\triangleleft$ *Sketch of a proof.* It is easy to show that the gradient formula (27) gives rise to the following (after changing variables $\mathbf{R} \ni x \longleftrightarrow \tau \in \mathbf{R}$) expression:

$$s_{12}(\tau;\omega) = \delta\gamma(l)(\lambda), \delta u\mid_{\lambda=\omega_j} = -\partial\gamma(l)(\lambda)/\partial\lambda\mid_{\omega_j}\delta\lambda_j/\delta u = \bar{c}_j q_j^2,$$

$$\bar{c}_j = -g\gamma(l)(\lambda)/d\lambda\mid_{\lambda=\omega_j}, \tag{46}$$

for all $j = \overline{0, N(\lambda)}$, where we have used such a property of the analytical Casimir functional $\gamma(l)(\lambda), \lambda \in \mathbf{C}$:

$$\gamma(l)(\omega_j) - 2 = 0, \tag{47}$$

that is $\gamma(l)(\omega_j) = 2 + r(\lambda)N(\lambda)\prod_{j=0}^{N(\lambda)}(\lambda - \omega_j)$ for some analytical functional $r(\lambda) \in \mathcal{D}(M(W))$, $\lambda \in \mathbf{C}$. Since the normalized component $s_{12}(\tau;\lambda)$ can be shown [17] to be a polynomial in $\lambda \in \mathbf{C}$ of the degree $N(\lambda)$ with the older coefficient to be the unity, we find the following interpolating result for this component on the manifold $M_N(W)$:

$$s_{12}(\tau;\lambda) = \prod_{j=0}^{N(\lambda)}\sum_{j=0}^{N(\lambda)}\frac{\bar{s}_j q_j^2}{\lambda - \omega_j}, \tag{48}$$

where $\bar{s}_j \in \mathcal{D}(M(W))$, $j = \overline{0, N(\lambda)}$, are some invariant coefficients to be determined further. Since $s_{12}(\tau;\lambda) \to \lambda^{N(\lambda)}$ if $\lambda \to \infty$, from (48) we obtain at once that $\sum_{j=0}^{N(\lambda)} \bar{s}_j q_j^2 = 1$. Normalizing the expression (48) by means of dividing it on the multiplier $\prod_{k=0}^{N(\lambda)}(\lambda - \omega_k)$, we can find from (48) that the main constraint condition $\sum_{j=0}^{N(\lambda)} q_j^2 = 1$ will be satisfied, if the norming invariant coefficients are equal to unity, that is, $\bar{s}_j = 1$ for all $j = \overline{0, N(\lambda)}$. In this case we can easily obtain the following Moser map:

$$s_{12}(\tau;\lambda) :\to s_{12}(q,p;\lambda) = \sum_{j=0}^{N(\lambda)}\frac{q_j^2}{\lambda - \omega_j}, \tag{49}$$

coinciding with the first expression in (44). Using further the monodromy matrix equation (26), the whole list of formulae in (44) is recovered successfully. This ends the sketch of a proof mentioned above. $\triangleright$

Since the functionals $det\,S(\tau;\lambda)$ and $Sp\,S^k(\tau;\lambda)$, $k \in \mathbf{Z}$, are invariant due to (26), we can construct the functional $\gamma(\lambda) := s^2(q,p;\lambda) + s_{12}(q,p;\lambda)s_{21}(q,p;\lambda) = -det\,S(q,p;\lambda)$, giving rise to a finite hierarchy of polynomial conservation laws as follows:

$$\gamma(\lambda) := 1 + \sum_{j=0}^{N(\lambda)} \gamma_j \frac{1}{\lambda - \omega_j}, \tag{50}$$

where for $j = \overline{0, N(\lambda)}$

$$\gamma_j = q_j^2 \left(\omega_j - \sum_{k=0}^{N(\lambda)} \omega_k q_k^2 - a_0 \right) + \sum_{k \neq j}^{N(\lambda)} \frac{(q_j p_k - q_k p_j)^2}{\omega_j - \omega_k}. \tag{51}$$

The corresponding Hamiltonian function (43) gets the following representation:

$$H = \frac{1}{2} \sum_{j=0}^{N(\lambda)} \omega_j \gamma_j \mid_{T^*(\mathbf{S}^{N(\lambda)}} . \tag{52}$$

It is also obvious that the set of functionals (51) is involutive on the adjoint space $T^*(\mathbf{S}^{N(\lambda)})$, that is

$$\{\gamma_j, \gamma_k\}_{\mathbf{S}^{(N)}} = 0 \tag{53}$$

for all $j, k = \overline{0, N(\lambda)}$. Thereby, the Liouville integrability of the dynamical system (41) on $T^*(\mathbf{S}^{N(\lambda)})$ is proved.

Now we are going to give a Lie-algebraic proof of the monodromy matrix equation (26) as a Lax-type equation for some moment map built via the scheme [1, 2].

Let be given the above considered loop algebra $\tilde{\mathcal{G}} = sl(2;\mathbf{R}) \otimes \mathbf{C}(\lambda, \lambda_{-1})$, $\lambda \in \mathbf{C}$. Since the monodromy matrix $S(\tau;\lambda) = grad\gamma(l) \in \tilde{\mathcal{G}}^*$ by construction, we can write the following its expansion in $\lambda \in \mathbf{C}$ at $|\lambda| \to \infty$:

$$S(\tau;\lambda) \mid_{M_N(W)} = S(q,p;\lambda) \to \sum_{j \geq 1} s_j \lambda^{-j} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S_2^{(12)} - \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv$$

$$-\lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \left(S_2 \lambda_{-2} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S_2^{(12)} \right) + \sum_{j \geq 1}^{j \neq 2} s_j \lambda^{-j} \in \delta\tilde{\mathcal{G}}_+^* + \xi, \tag{54}$$

where the element $\xi = \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}$ must belong to $\delta\tilde{\mathcal{G}}_+^0 \cap [\delta\tilde{\mathcal{G}}_-, \delta\tilde{\mathcal{G}}_-]^0$ and be interpreted as an infinitesimal character of a Lie subalgebra $\delta\tilde{\mathcal{G}}_- \subset \tilde{\mathcal{G}}$,

$$\begin{aligned} \delta\tilde{\mathcal{G}}_+ : &= \left\{ \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bar{X}_2^{(21)} \oplus \bar{X}_2 \lambda^2 \right) + \sum_{j \geq 1}^{(j \neq 2)} X_j \lambda^j \right\}, \\ \delta\tilde{\mathcal{G}}_- : &= \left\{ \left(\bar{Y}_2 \lambda^{-2} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \bar{Y}_2^{(12)} \right) + \sum_{j \geq 1}^{(j \neq 2)} Y_j \lambda^j \right\}, \end{aligned} \tag{55}$$

with $\bar{X}_2, \bar{Y}_2 \in \mathcal{G}$ chosen in some form. Therefore, we have the following direct sum splitting of the loop Lie algebra $\delta\tilde{\mathcal{G}}$:

$$\delta\tilde{\mathcal{G}} = \delta\tilde{\mathcal{G}}_+ \oplus \delta\tilde{\mathcal{G}}_-, \quad \delta\tilde{\mathcal{G}}^* = \delta\tilde{\mathcal{G}}^*_+ \oplus \delta\tilde{\mathcal{G}}^*_-, \tag{56}$$

where $\delta\tilde{\mathcal{G}}^*_+ \cong \delta\tilde{\mathcal{G}}_-$, $\delta\tilde{\mathcal{G}}^*_- \cong \delta\tilde{\mathcal{G}}_+$, the conjugation "*" being taken with respect to the following scalar product on $\tilde{\mathcal{G}}$:

$$(a(\lambda), b(\lambda))_0 := res_{\lambda=\infty}\frac{1}{\lambda}Sp(a(\lambda)b(\lambda)) \tag{57}$$

for any $a(\lambda)$, $b(\lambda) \in \delta\tilde{\mathcal{G}}$, $\lambda \in \mathbf{C}$, which ensures the wanted property $(\delta\tilde{\mathcal{G}}^*_+)^* := \delta\tilde{\mathcal{G}}_+$, that is $\delta\tilde{\mathcal{G}}_+$ be a loop Lie subalgebra of the loop Lie algebra $\delta\tilde{\mathcal{G}}$. We notice here that this is the unique way to do this suitably. If the splitting (56) is made, we can convince ourselves that the element $\xi = \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}$ indeed is the infinitesimal character of the above built loop Lie subalgebra $\delta\tilde{\mathcal{G}}_-$, that is $\xi \in \delta\tilde{\mathcal{G}}^0 \cap [\delta\tilde{\mathcal{G}}_-, \delta\tilde{\mathcal{G}}_-]^0 \in \delta\tilde{\mathcal{G}}^*$. As a result, we have the following Lax-type representation of the Neumann-type oscillatory dynamical system (41) stemming from (36), (39), (44), (45) and (12):

$$\frac{dS(q,p;\lambda)}{d\tau} = [l(q,p;\lambda), S(q,p;\lambda)], \tag{58}$$

where for all $\lambda \in \mathbf{C}$, $(q,p)^\tau \in T^*(\mathbf{S}^{N(\lambda)})$

$$l(q,p;\lambda) := \begin{pmatrix} 0 & 1 \\ & \\ u(q,p) + \lambda v(q,p) - \lambda^2 & 0 \end{pmatrix},$$

$$\begin{aligned} S(q,p;\lambda) : \ = \ & \sum_{j=0}^{N(\lambda)} \frac{1}{(\lambda - \omega_j)} \begin{pmatrix} -q_j p_j & q_j^2 \\ & \\ -p_j^2 & q_j p_j \end{pmatrix} + \\ & \left(-\lambda + \sum_{j=0}^{N(\lambda)} \omega_j q_j^2 + a_0\right) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{59}$$

where

$$u(q,p) = -\sum_{j=0}^{N(\lambda)} [(dq_j/d\tau)^2 + \omega_j^2 q_j^2 - 2\omega_j q_j^2 \sum_{j=0}^{N(\lambda)} \omega_j q_j^2],$$

$$v(q,p) = a_0 + 2\sum_{j=0}^{N(\lambda)} \omega_j q_j^2,$$

$a_0 \in \mathbf{R}$-arbitrary.

A sequel of this paper are we going to devote to a construction of some dual moment map from the canonical symplectic space $T^*(\mathbf{R}^{N(\lambda)+1})$ to the dual space of the above involved loop Lie algebra $\tilde{\mathcal{G}}_+$. As it would be shown , this moment map generates an infinite hierarchy of Lax-type integrable dynamical systems on the adjoint space $T^*(\mathbf{S}^{N(\lambda)}) \subset T^*(\mathbf{R}^{N(\lambda)+1})$, one of which just coincides with that generated by the representation of this construction, and gives rise to the infinite hierarchy of so-called Neumann-Rosochatius-type dynamical systems on $T^*(\mathbf{S}^{N(\lambda)})$, before studied thoroughly in [6,16] by means of essentially other methods.

We are going to use effectively the above introduced loop Lie algebra $\tilde{\mathcal{G}}$, allowing the direct sum splitting (56). Let a matrix $Q \in gl(n; \mathbf{R})$, $n \in \mathbf{Z}_+$ is arbitrary, be some fixed matrix with the spectrum being contained inside a large fixed disc D centered at $\mathbf{C} \ni \lambda = 0$. Thereby, the group $\tilde{G}_+ := sl(2; \mathbf{R}) \otimes \mathbf{C}(\lambda)$, whose the loop Lie algebra is $\tilde{\mathcal{G}}_+$, coincides with that of holomorphic matrix functions in $\lambda \in \mathbf{C}$. Let us consider a matrix manifold $\tilde{M} = M_{n,2} \times M_{n,2}$ with $n \geq 2 \in \mathbf{Z}_+$, where $M_{n,2}$ denotes the space of $(n \times 2)$-matrices over the field of complex numbers. This manifold carries a natural symplectic structure

$$\Omega^{(2)}(F, G) := Sp(dF \wedge dG^\tau), \tag{60}$$

where $(F, G) \in \tilde{M}$. The symplectic structure (60) allows a right symplectic group action on the manifold $\tilde{M}$

$$g(\lambda) : (F, G) \longrightarrow (F_g, G_g), \tag{61}$$

where, by definition,

$$\begin{aligned} F_g : &= res_{\lambda \in D} \left(\frac{1}{\lambda - Q} F g^{-1}(\lambda) \right), \\ G_g^\tau : &= res_{\lambda \in D} \left(g(\lambda) G^\tau \frac{1}{\lambda - Q} \right) \end{aligned} \tag{62}$$

for any element $g(\lambda) \in \tilde{G}_+$, $\lambda \in \mathbf{C}$. It is an easy exercise to prove, using the Hilbert's resolvent identity, that the actions (61) , (62) satisfy the standard group action properties and the symplectic structure (60) be invariant indeed. Now we are going to imbed the group action (61) of the group $\tilde{G}_+$ on the manifold $\tilde{M}$ into some group action space of a specially constructed deformed group $\delta\tilde{G}_+ := \exp(\delta\tilde{\mathcal{G}}_+)$ over the loop Lie algebra basis $\delta\tilde{\mathcal{G}}_+$. To do this, let there be given an element $\bar{X}(\lambda) \in \delta\tilde{\mathcal{G}}_+$ as follows:

$$\bar{X}(\lambda) := \bar{X}_2^{(12)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \bar{X}_2 \lambda^2 + \sum_{j \geq 1}^{(j \neq 2)} X_j \lambda^j, \tag{63}$$

where $X_2, X_j \in sl(2; \mathbf{R})$, $j - 1 \in \mathbf{Z}_+$, are arbitrary. Over the basis of elements (65) we can construct an isomorphic map $\delta : \tilde{\mathcal{G}}_+ \to \delta\tilde{\mathcal{G}}_+$, which acts via the rule: if an element $X(\lambda) := \sum_{j \in \mathbf{Z}_+} X_{j+1} \lambda^j \in \tilde{\mathcal{G}}_+$, the result will be

$$\delta : \tilde{\mathcal{G}}_+ \ni X(\lambda) \to \bar{X}(\lambda) \in \delta\tilde{\mathcal{G}}_+ \tag{64}$$

where $\bar{X}_2^{(12)} := 1/2 X_2^{(12)}$ by definition. Thereby, we can compute now easily the induced moment map of the group action (61). The following theorem is valid.

Theorem 2. *The $\delta\tilde{\mathcal{G}}_+$-action (61) is Hamiltonian, with the equivariant moment map $S_Q : \tilde{M} \to \delta\tilde{\mathcal{G}}_+^*$, where*

$$S_Q(F, G; \lambda) = G^\tau (\lambda - Q)^{-1} F \oplus res_{\lambda=0} (G^\tau \lambda^{-1} (\lambda - Q)^{-1} F)^{(12)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{65}$$

Proof. ◁ Since the action (61) is the cotangent lift of the linear actions (64) on the matrix manifold $M_{n,2}$, it is obviously Hamiltonian. To compute the corresponding moment map of this action (61), we need to find the Hamiltonian function of some one-parameter group action (61) on the manifold $\tilde{M}$ by an element $g(t;\lambda) := \exp[tX(\lambda)] \in \tilde{G}_+$, $X(\lambda) \in \tilde{\mathcal{G}}_+$, $t \in \mathbf{R}$

$$\begin{aligned} \frac{dF_g}{dt} &= -res_{\lambda\in D}\left(\frac{1}{\lambda-Q}FX(\lambda)\right) := -\frac{\partial H_{\bar{X}}(F,G)}{\partial G^\tau}, \\ \frac{dG_g}{dt} &= res_{\lambda\in D}\left(X(\lambda)G^\tau\frac{1}{\lambda-Q}\right) := \frac{\partial H_{\bar{X}}(F,G)}{\partial F}. \end{aligned} \tag{66}$$

From (66) we can easily find that the Hamiltonian function $H_X(F,G) \in \mathcal{D}(\tilde{M})$ takes the form:

$$\begin{aligned} H_X(F,G) &= res_{\lambda\in D}\left(G^\tau\frac{1}{\lambda-Q}FX(\lambda)\right) \equiv \\ &\left(G^\tau\frac{1}{\lambda-Q}F,\bar{X}\right)_0 := (S_Q(F,G;\lambda),\bar{X})_0, \end{aligned} \tag{67}$$

where the element $\bar{X} \in \delta\tilde{\mathcal{G}}_+$ and $S_W(F,G;\lambda) \in \delta\tilde{\mathcal{G}}_+^*$ are defined by (64) and (65). As a result of (67) the exact form of the moment map (65) is proved. ▷

To use the result above, we need further to describe a hierarchy of Casimir functionals on the adjoint space $\delta\tilde{\mathcal{G}}_+^*$. We have by definition that the functional $\delta\gamma \in \mathcal{D}(\tilde{\mathcal{G}}^*)$ is a Casimir one if $[\nabla\gamma(S),S] = 0$ for all $S \in \tilde{\mathcal{G}}^*$. This condition is obviously satisfied for

$$\gamma_m := res_{\lambda\in D} SpS^m, \quad n \in \mathbf{Z}_+. \tag{68}$$

In the case when $S(\lambda) := S_Q(F,G;\lambda) \in \delta\tilde{\mathcal{G}}_+^*$, the formula (68) gives only one nontrivial invariant functional at $m = 1$. To overcome this difficulty we must use the following theorem.

Theorem 3. (Adler/Kostant/Symes [9]): *Let us choose an infinitesimal character $\xi \in \delta\tilde{\mathcal{G}}^*$ of the subalgebra $\delta\tilde{\mathcal{G}}_-$, that is, it belongs to the subspace $\delta\tilde{\mathcal{G}}_+^0 \cap [\delta\tilde{\mathcal{G}}_-,\delta\tilde{\mathcal{G}}_-]^0$:*

$$(\xi,[\delta\tilde{\mathcal{G}}_-,\delta\tilde{\mathcal{G}}_-])_0 = 0 = (\xi,\delta\tilde{\mathcal{G}}_+^0)_0. \tag{69}$$

Then:

a) the Hamiltonian vector fields with respect to the standard Lie-Poisson bracket on $\delta\tilde{\mathcal{G}}_+^*$, built of the restriction to $\xi + \delta\tilde{\mathcal{G}}_+^*$ of invariants $\gamma_m \in \mathcal{D}(\tilde{\mathcal{G}}^*)$, $m \in \mathbf{Z}_+$, are given by

$$ds/dt = [s+\xi, \nabla\gamma_m(s+\xi)_+], \tag{70}$$

where $s \in \delta\tilde{\mathcal{G}}_+^*$,$(...)_+$ is the projection on the $\tilde{\mathcal{G}}_+$ Lie subalgebra;

b) the set of all restrictions to $\xi + \delta\tilde{\mathcal{G}}_+$ of invariant functions (68) is involutive with respect to the standard Lie-Poisson bracket on $\delta\tilde{\mathcal{G}}_+^*$.

Proof. The proof of this theorem is not complex and is contained in [18, 19].

The following element of $\delta\tilde{\mathcal{G}}^*$

$$\begin{pmatrix} 0 & 0 \\ \lambda - a_0 & 0 \end{pmatrix} \tag{71}$$

is a one-point orbit of the loop algebra $\delta\tilde{\mathcal{G}}$ with conditions (69) being satisfied. This means the following lemma is true.

Lemma. *The element $\xi \in \delta\tilde{\mathcal{G}}^*$ (71) is an infinitesimal character of the loop Lie subalgebra $\delta\tilde{\mathcal{G}}_-$, the direct sum splitting of $\delta\tilde{\mathcal{G}}$ (56) being realized.*

Using the above Lemma and Theorems 2 and 3, we can formulate the following important result as

Theorem 4. *The Neumann-type dynamical system (40) takes the Lax type representation as follows:*

$$\frac{dS_Q}{d\tau} = \left[S_Q + \xi, \left\{ \frac{det(\lambda - Q)}{\lambda^{N(\lambda)}} (\xi + S_Q) \right\}_+ \right], \tag{72}$$

where we have made over $(F, G) \in \tilde{M}$ the following reduction to the manifold $T^*(\mathbf{R}^{N(\lambda)+1})$:

$$F := \begin{pmatrix} q_0, & q_1, & ..., & q_{N(\lambda)} \\ & & & \\ p_0, & p_1, & ..., & p_{N(\lambda)} \end{pmatrix}^\tau, \tag{73}$$

$$G := F\sigma_1 = \begin{pmatrix} p_0, & p_1, & ..., & p_{N(\lambda)} \\ & & & \\ -q_0, & -q_1, & ..., & -q_{N(\lambda)} \end{pmatrix}^\tau, \quad \sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and used the following Casimir-type Hamiltonian function $H \in \mathcal{D}(\tilde{\mathcal{G}}_+^*)$:

$$H = \frac{det(\lambda - Q)}{2\lambda^{N(\lambda)}} (\xi + S_Q)^2 \tag{74}$$

to build the evolution (72).

Proof. $\triangleleft$ From (65), (71) and (73) for $s \to S_Q \in \tilde{\mathcal{G}}_+^*$ and $n := N(\lambda) \in \mathbf{Z}_+$, we herewith obtain that the extended element $S_Q + \xi \equiv S(q, p; \lambda)$, which is defined by (44) and (45), and the suitably reduced manifold $\tilde{M}$ is diffeomorphic to the manifold $T^*(\mathbf{R}^{N(\lambda)+1})$, before built in this chapter. Due to equation (26), we can easily find also that the Hamiltonian function (43) has the representation (74), that is

$$H = res_{\lambda \in D} \left\{ \frac{det(\lambda - Q)}{2\lambda^{N(\lambda)}} SpS^2(q, p; \lambda) \right\}, \tag{75}$$

the Lax element $l[q, p; \lambda] \in \tilde{\mathcal{G}}_+$ being found in accordance with (72) as follows:

$$l[q, p; \lambda] = \left\{ \frac{det(\lambda - Q)}{\lambda^{N(\lambda)}} S(q, p; \lambda) \right\}_+ . \tag{76}$$

The latter proves the Theorem. $\triangleright$

The above stated Theorem 4 is easily generalized to the one, giving the Lax-type representation of the so-called Neumann-Rosochatius-type dynamical system on the cotangent space to the sphere $\mathbf{S}^{N(\lambda)}$, previously studied in [6]:

$$dp_j/d\tau = -\omega_j^2 q_j + v(q, p) q_j \omega_j + \alpha_j^2 q_j^{-3}, dq_j/d\tau = p_j \tag{77}$$

where $v(q,p) := 2\sum_{j=0}^{N(\lambda)} \omega_j q_j^2 + \alpha_0$, α_0 is an arbitrary constant, $(q,p)^\tau \in T^*(\mathbf{S}^{N(\lambda)})$, that is, $\sum_{j=0}^{N(\lambda)} q_j^2 = 1, \sum_{j=0}^{N(\lambda)} q_j p_j = 0$. To make this generalization, we need to reduce the loop Lie algebra $\tilde{\mathcal{G}} = sl(2;\mathbf{C}) \otimes \mathbf{C}(\lambda,\lambda^{-1})$ to $\tilde{u}(1,1) := u(1,1) \otimes \mathbf{C}(\lambda)$, $\lambda^{-1}, \lambda \in \mathbf{C}$, as this has been done in [2]. Let us consider a space $M'_N(W) \subset M_{N(\lambda),2} \times M_{N(\lambda),2}$ as a symplectic subspace given by

$$M'_N(W) := \left\{(F,G) \in M_N(W) : G = \bar{F}\sigma_1\right\}, \tag{78}$$

where the bar over the matrix F means the complex conjugation.

The moment map (65) restricted to the manifold $M'_N(W)$ (80) gives a moment map $S_Q(F;\lambda) \in \delta\tilde{u}(1,1)^*_+$, where, by definition, we have a direct sum splitting of the deformed loop Lie algebra $\delta\tilde{u}(1,1)$:

$$\delta\tilde{u}(1,1) := \delta\tilde{u}(1,1) \oplus \delta\tilde{u}(1,1), \tag{79}$$

found in a natural way from the splitting (56). The restricted moment map $S_Q(F,\lambda)$ is invariant under the action of the group $\bigotimes_{N(\lambda)} U(1) = U(1) \times U(1) \times ... \times U(1) - N(\lambda)$ times) contained in $Gl(N(\lambda);\mathbf{C})$.

Performing the usual Marsden-Weinstein reduction of $M'_N(W)$ to $M_N(W)$ by this $\bigotimes_{N(\lambda)} U(1)$-action at the $\bigotimes_{N(\lambda)} U(1)^*$-moment map value $\sqrt{-2}(\alpha_0,\alpha_1,..,\alpha_{N(\lambda)}) \in \bigotimes_{N(\lambda)} U(1)^*$ gives the injective Poisson map $S : M_N(W) \to \delta\tilde{u}(1,1)^*_+$, $M_N \simeq T^*(\mathbf{R}^{N(\lambda)+1})$, given by

$$\begin{aligned} S_Q(q,p;\lambda) : &= \sum_{j=0}^{N(\lambda)} \frac{1}{\lambda - \omega_j} \begin{pmatrix} -q_j p_j & q_j^2 \\ -p_j^2 & q_j p_j \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \sum_{j=0}^{N(\lambda)} \omega_j q_j^2 - \\ & \sum_{j=0}^{N(\lambda)} \frac{1}{\lambda - \omega_j} \left\{ \alpha_j \sqrt{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\frac{\alpha_j^2}{q_j^2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}. \end{aligned} \tag{80}$$

The latter formula (80) gives rise, due to the general Theorem 3, to the following theorem to be true.

Theorem 5. *The Neumann-Rosochatius oscillatory dynamical system (77) on the symplectic manifold $T^*(\mathbf{S}^{N(\lambda)})$ is a Hamiltonian completely integrable one with a Lax-type representation, being given in the form (72) with the element $S_Q := S_Q(q,p;\lambda) \in \delta\tilde{u}^*_+(1,1)$ defined by (80).*

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