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(p, q)-Analog of Two-Dimensional Conformal Field Theory. The Ward Identities and Correlation Functions

I.M. BURBAN

Bogolyubov Institute for Theoretical Physics, 252143 Kyiv, Ukraine

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Abstract

A (p, q)-analog of two-dimensional conformally invariant field theory based on the quantum algebra $U_{pq}(su(1, 1))$ is proposed. The representation of the algebra $U_{pq}(su(1, 1))$ on the space of quasi-primary fields is given. The (p, q)-deformed Ward identities of conformal field theory are defined. The two- and three-point correlation functions of quasi-primary fields are calculated.

1 Introduction

Two-dimensional conformal field theory is closely related to the theory of quantum groups. Different generalizations of conformal field theory are based on different deformations of the symmetry algebra. One of them modifies the entire structure of the theory by means of a deformation of the full Virasoro algebra [1, 2]. Since deformations preserving the Hopf algebra structure of the Virasoro algebra are still unknown, the papers [1, 2] are restricted by the q-deformations of its subalgebra $su(1, 1)$.

The q-deformations of vertex operators and their correlation functions have been studied in [3].

In this paper we propose a (p, q)-analog of two-dimensional conformal field theory based on the (p, q)-deformation of the subalgebra $su(1, 1)$ of the Virasoro algebra. The quantum algebra $U_{pq}(su(1, 1))$ has been constructed in [4] and its representations have been classified in [5].

In the next section we realize the representation of the algebra $U_{pq}(su(1, 1))$ on the space of quasi-primary fields. The (p, q)- deformed $su(1, 1)$ Ward identities are obtained in Section 3. In Section 4 the two- and three-point correlation functions of the quasi-primary fields are determined.

2 Realization of the representation

The space of states of conformal field theory is an inner product space carrying the representation of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n+1)\delta_{m+n,0}, \quad n = 1, 2, \dots \tag{1}$$

The properties of the correlation functions of the theory are determined by the Ward identities for the subalgebra $su(1,1)$ of the Virasoro algebra (1)

$$[E_0, E_{+1}] = E_{+1}, \quad [E_0, E_{-1}] = -E_{-1}, \quad [E_{-1}, E_{+1}] = E_0, \tag{2}$$

where $E_0 = L_0, E_{-1} = -L_{-1}, E_{+1} = -L_{+1}$. The universal enveloping algebra $U(su(1,1))$ of the Lie algebra (2) admits the Hopf algebra structure, in particular, the operation of the comultiplication is

$$\Delta(E_n) = E_n \otimes 1 + 1 \otimes E_n, \quad n = 0, \pm 1. \tag{3}$$

The homomorphism $\Delta : su(1,1) \to su(1,1) \otimes su(1,1)$ can be extended to the homomorphism $su(1,1) \to \otimes_{i=1}^{N} \otimes su(1,1)$ by the formula

$$\Delta^N(E_n) = (\Delta \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id}) \dots (\Delta \otimes \mathrm{id})\Delta(E_n), \qquad n = 0, \pm 1. \tag{4}$$

There exist various deformations of the universal enveloping algebra $U(su(1,1))$ preserving the Hopf algebra structure. As such it is the (p,q)-deformation $U_{pq}(su(1,1))$ of this algebra. The generators K_{+1}, K_{-1}, K_0 of the quantum algebra $U_{pq}(su(1,1))$ satisfy the following commutation relations [4]

$$\begin{aligned} &[K_0, K_{+1}] = K_{+1}, \qquad [K_0, K_{-1}] = -K_{-1}, \\ &[K_{-1}, K_{+1}]_{pq} = K_{-1}K_{+1} - qp^{-1}K_{+1}K_{-1} = [2K_0]_{pq}, \end{aligned} \tag{5}$$

where $[a]_{pq} = \frac{q^a - p^{-a}}{q - p^{-1}}$, and p, q are complex parameters. The algebra (5) admits the Hopf algebra structure, in particular, the operation of the comultiplication is

$$\Delta(K_{\pm 1}) = q^{K_0} \otimes K_{\pm 1} + K_{\pm 1} \otimes p^{-K_0}, \quad \Delta(K_0) = K_0 \otimes 1 + 1 \otimes K_0. \tag{6}$$

The representation of the commutation relations (5) on the space F of functions $f(z)$ is defined by

$$\begin{aligned} K_{+1}f(z) &= z\frac{q^{2h}f(zq) - p^{-2h}f(zp^{-1})}{q - p^{-1}}, \\ K_{-1}f(z) &= \frac{1}{z}\frac{f(zq) - f(zp^{-1})}{q - p^{-1}}, \\ \mathcal{K}_q f(z) &\stackrel{\text{def}}{=} q^{K_0}f(z) = q^h f(qz), \end{aligned} \tag{7}$$

where h is the conformal dimension of the quasi-conformal field.

3 The (p,q)-deformed conformal Ward identities

A quasi-primary field $\phi_h(z)$ with the conformal dimension h is the field transforming under $SU_{pq}(su(1,1))$ as

$$[\hat{K}_n, \phi_h] = \{z^n[(n+1)h]_{pq}\phi_h(zq) + p^{-(n+1)h}z^{n+1}(D_{pq}\phi_h)(z)\}\hat{\mathcal{K}}_p^{-1}, \quad n = \pm 1,$$

$$\hat{\mathcal{K}}_q\phi_h(z)\hat{\mathcal{K}}_p^{-1} = q^h\phi_h(zq). \tag{8}$$

The commutator on the left-hand side of the first equality in (8) is defined as

$$[A, \phi_h(z)] = A\phi_h(z) - \hat{\mathcal{K}}_q\phi_h(z)\hat{\mathcal{K}}_q^{-1}A \tag{9}$$

with $A \in U_{pq}(su(1,1))$. The formulae (8) and (9) at $p = 1$ coincide with the formulae (7) and (8) of ref. [2], and at $p = q = 1$ we obtain

$$[\hat{K}_n, \phi_h(z)] = z^n[z\partial_z + h(n+1)]\phi_h(z), \qquad n = 0, \pm 1, \tag{10}$$

that is the transformation law of primary fields of conformal field theory. The structure of the commutator (9) is equivalent to the coproduct given by (6), D_{pq} in (8) is a (p,q)-derivative [6]

$$D_{pq}f(z) = \frac{1}{z}\frac{f(qz) - f(p^{-1}z)}{q - p^{-1}} \tag{11}$$

satisfying

$$D_{pq}[f_1(z)f_2(z)] = (D_{pq}f_1)(z)f_2(qz) + f_1(p^{-1}z)(D_{pq}f_2)(z). \tag{12}$$

The $U_{pq}(su(1,1))$ invariant vacuum $|0\rangle$, $\hat{K}_{\pm1}|0\rangle = 0$, $\hat{\mathcal{K}}_q\||0\rangle = |0\rangle$, and quasi-primary fields $\phi_{h_1}(z), \phi_{h_2}(z), \ldots, \phi_{h_N}(z)$ of the conformal weights $h_1, h_2, \ldots, h_N$, respectively, define the correlation functions

$$\langle\phi_1(z_1)\ldots\phi_N(z_N)\rangle_{pq} = \langle 0|\phi_{h_1}(z_1)\ldots\phi_{h_N}(z_N)|0\rangle_{pq}. \tag{13}$$

Using the commutation relation (8), (9) and the $U_{pq}(su(1,1))$ invariance of the vacuum, we obtain the equations which provide the $U_{pq}(su(1,1))$ invariance of the correlation functions

$$0 = \langle\hat{K}_n\phi_1(z_1)\ldots\phi_N(z_N)\rangle = \sum_{j=1}^{N} q^{h_1+h_2+\ldots+h_{j-1}}p^{-h_{j+1}-\ldots-h_N}\times$$

$$\langle\phi_1(qz_1)\ldots\phi_{j-1}(qz_{j-1})\hat{\phi}_i(z_i)\phi_{j+1}(p^{-1}z_{j+1}\ldots\phi(p^{-1}z_N))\rangle_{pq}, \tag{14}$$

$$\langle\hat{\mathcal{K}}_q\phi_1(z_1)\ldots\phi_N(z_N)\rangle_{pq} = q^{h_1+h_2+\ldots+h_N}\langle\phi_1(qz_1)\ldots\phi_N(qz_N)\rangle_{pq}, \tag{15}$$

where $\hat{\phi}(z) = \{[(n+1)h]_{pq}z^n\phi(qz) + z^{n+1}D_{pq}\phi(z)\}\hat{\mathcal{K}}_p^{-1}$, $\quad n = \pm 1$. With the help of (6) and (7), the equation (14) can be rewritten as

$$\overbrace{(\Delta\otimes \mathrm{id}\otimes\ldots\otimes \mathrm{id})}^{N-1}\overbrace{(\Delta\otimes \mathrm{id}\otimes\ldots\otimes \mathrm{id})}^{N-2}\ldots(\Delta\otimes \mathrm{id})\times$$

$$\Delta(K_{\pm 1})\langle\phi_1(z_1)\phi_2(z_2)\ldots\phi_N(z_N)\rangle_{pq}=0. \tag{16}$$

The equations (15), (16) define the (p,q)-deformation of the $su(1,1)$ Ward identities of conformally invariant field theory.

4 Two- and three-point correlation functions

As in the classical case, the two- and three-point correlation functions can be obtained as solutions of the Ward identities (15), (16). In particular, with the help of (6) the identities (15), (16) can be rewritten as

$$\Delta(K_{\pm 1})\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}=(q^{K_0}\otimes K_{\pm 1}+K_{\pm 1}\otimes p^{-K_0})\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}=0,$$
$$\Delta(\mathcal{K}_p)\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}=\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}. \tag{17}$$

From (17) we obtain the following set of the difference equations

$$\frac{p^{-h_2}}{z_1}\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\rangle_{pq}-(\frac{p^{-h_2}}{z_1}-\frac{q^{h_1}}{z_2})\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\rangle_{pq}-$$
$$\frac{q^{h_1}}{z_2}\langle\phi_i(qz_1)\phi_2(qz_2)\rangle_{pq}=0,$$
$$p^{-2h_1-h_2}z_1\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\rangle_{pq}-(q^{2h_1}p^{-h_2}z_1-q^{h_1}p^{-2h_2}z_2)\times \tag{18}$$
$$\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\rangle_{pq}-q^{h_1+2h_2}z_2\langle\phi_i(qz_1)\phi_j(qz_2)\rangle_{pq}=0,$$
$$q^{h_i+h_j}\langle\phi_i(qz_1)\phi_j(qz_2)\rangle_{pq}=\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}.$$

The set of equations (18) is consistent and admits a solution if and only if the two conformal weights $h_1$and h_2 are equal: $h_1=h_2=h$. A solution of equations (18) can be obtained with the following ansatz

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle_{pq}=C(p,q)z^{-a}{}_1\phi_0^{pq}(a;(pq)^{\alpha}z_2/z_1), \tag{19}$$

where

$${}_1\phi_0^{pq}(a;z_1/z_2)=\frac{((pq)^a z_2/z_1;p,q)_\infty}{(z_2/z_1;p,q)_\infty} \tag{20}$$

is the (p,q)-hypergeometric function [7] with

$$(z_2/z_1;p,q)_\infty=\prod_{l=0}^{\infty}(p^{-l}-q^l z_2/z_1). \tag{21}$$

The function (20) is a solution of the following (p,q)-difference equation

$$(1-(pq)^{\alpha-1}z_2/z_1){}_1\phi_0^{pq}(a;(pq)^{\alpha-1}z_2/z_1)=$$
$$(1-(pq)^{a+\alpha-1}z_2/z_1){}_1\phi_0^{pq}(a;(pq)^{\alpha}z_2/z_1). \tag{22}$$

Taking into account formula (22), the solution (19) of equations (18) can be written as

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle_{pq} = C(p,q)z_1^{-2h}\phi_0^{pq}(2h;(pq)^{1-h}z_2/z_1) =$$

$$C(p,q)z^{-2h}\frac{((pq)^{1+h}z_2/z_1;p,q)_\infty}{((pq)^{1-h}z_2/z_1;p,q)_\infty}. \tag{23}$$

In [8, 9] the solution (23) has been represented in a different form.

The limiting cases of equations (18) have been studied in [1] and [2]. If we put $p=1$ in (18) and (19), we obtain equations (18)–(20) and its solution (21) from [2]. The set of equations (2.1)–(2.3) from [1] is obtained, if we carry out in (18) the replacement $p=q\to q^{-1}$. By virtue of (22), the function $\phi(z) = {}_1\phi_0^{q^{-1},q^{-1}}(2h;q^{-2(1-h)}z)$, $z=z_2/z_1$, satisfies the difference equation

$$\phi(zq^2) = \frac{1-zq^{-2h}}{1-zq^{2h}}\phi(z), \tag{24}$$

which admits a unique solution in $C[[z]]$:

$$\phi(z) = \frac{(zq^{2h};q^2)_\infty}{(zq^{-2h};q^2)_\infty}. \tag{25}$$

Hence, the solution (25) coincides with the function $G_q(z_2/z_1/2h;-2h)$ (2.7) of ref. [1].

The (p,q)-deformed Ward identities (16) for the three-point correlation function $\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq}$ can be rewritten as

$$(K_{+1}\otimes p^{-K_0}\otimes p^{-K_0} + q^{K_0}\otimes K_{+1}\otimes p^{-K_0} + q^{K_0}\otimes q^{K_0}\otimes K_{+1})\times$$

$$\langle\phi_i(z_1)\phi_(z_2)\phi_k(z_3)\rangle_{pq} = 0,$$

$$(K_{-1}\otimes p^{-K_0}\otimes p^{-K_0} + q^{K_0}\otimes K_{-1}\otimes p^{-K_0} + q^{K_0}\otimes q^{K_0}\otimes K_{-1})\times$$

$$\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq} = 0, \tag{26}$$

$$q^{h_1+h_2+h_3}\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = \langle\phi_i(z_1)\phi_j(z_3)\rangle_{pq}.$$

The set of equations (26) reduces to the following set of difference equations

$$p^{-2h_1-h_2-h_3}z_1\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq} -$$

$$(p^{-h_2-h_3}q^{2h_1}z_1 - p^{-2h_2-h_3}q^{h_1}z_2)\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq} -$$

$$(p^{-h_3}q^{h_1+2h_2}z_2 - p^{-2h_3}q^{h_1+h_2}z_3)\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(p^{-1}z_3)\rangle_{pq} -$$

$$q^{h_1+h_2+2h_3}z_3\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = 0,$$

$$\frac{1}{z_1}p^{-h_2-h_3}\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z)\phi_k(p^{-1}z_3)\rangle_{pq} - \tag{27}$$

$$\Big(\frac{1}{z_1}p^{-h_2-h_3} - \frac{1}{z_2}p^{-h_3}q^{h_1}\Big)\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq} -$$

$$\Big(\frac{1}{z_2}p^{-h_3}q^{h_1} - \frac{1}{z_2}q^{h_1+h_2}\Big)\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(p^{-1}z_3)\rangle_{pq} -$$

$$\frac{1}{z_3}q^{h_1+h_3}\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = 0,$$

$$q^{h_1+h_2+h_3}\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = \langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq}.$$

This set of equations is consistent and completely defines the three-point correlation function of quasi-primary fields

$$\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq} = C_{ijk}(p,q) z_1^{-\gamma_{12}^3-\gamma_{31}^2} z_2^{-\gamma_{23}^1} \times \tag{28}$$

$$ {}_1\phi_0^{pq}(\gamma_{12}^3; (pq)^{1-h_1} z_2/z_1)\ {}_1\phi_0^{pq}(\gamma_{23}^1; (pq)^{1-h_2} z_3/z_2)\ {}_1\phi_0^{pq}(\gamma_{31}^2; (pq)^{1-h_1+h_2} z_3/z_1)$$

with $\gamma_{ij}^k = h_i + h_j - h_k$. The three-point correlation function (28) in the limiting cases $p = 1$ and $p = q \to q^{-1}$ coincides with the ones of [1] and [2].

References

[1] Bernard D., LeClair A., *Phys. Lett. B*, 1989, V.227, 417.

[2] Matsuzaki T., Suzuki T., *Phys. Lett. B*, 1992, V.33, 296.

[3] Frenkel I.B., Reshetikhin N.Y., *Commun. Math. Phys.*, 1992, V.146, 1.

[4] Chakrabarti R., Jagannathan R., *J. Phys. A: Math. Gen.*, 1991, V.24, L711.

[5] Klimyk A.U., *J. Phys. A: Math. Gen.*, 1993, V.26, 7503.

[6] Burban I.M., *Phys.Lett. B*, 1993, V.319, 485.

[7] Burban I.M., Klimyk A.U., *Integral transformations and special functions*, 1994, V.2, 15.

[8] Burban I.M., (p,q)-deformation of $su(1,1)$ conformal Ward identities and correlation functions, Proceedings of the Eleventh International Hutsulian Workshop on Group-Theoretical Methods in Physics, October 25–31, 1993, Rakhiv, Ukraine, Naukova Dumka Publ., Kyiv, 1994.

[9] Chung W.-S., *Nuovo Cimento B*, 1994, V.109, 599.