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Representations of $*$-algebras and dynamical systems

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Introduction

1. Consider a $*$-algebra $\mathcal{A}$ generated by self-adjoint elements $a_1, \dots, a_n$ ($a_j = a_j^*$, $j = 1, \dots, n$) and the relations

$$P_k(a_1, \dots, a_n) = 0 \qquad (k = 1, \dots, m). \tag{1}$$

Here $P_k(\cdot)$ are polynomials in the non-commuting variables $a_1, \dots, a_n$ over $\mathbf{C}$ such that $P_k^*(\cdot) = P_k(\cdot)$. In other words, $\mathcal{A}$ is a quotient of the free $*$-algebra $\mathbf{C}\langle a_1, \dots, a_n\rangle$ generated by the self-adjoint elements $a_1, \dots, a_n$ with respect to the two-sided ideal generated by the relations (1).

Representations ot $\mathcal{A}$ ($*$-homomorphisms $\pi\colon \mathcal{A} \to L(H)$ of the $*$-algebra $\mathcal{A}$ into a $*$-algebra $L(H)$ of bounded operators on a separable Hilbert space H or into a $*$-algebra of unbounded operators) are of interest both from mathematical point of view and for their physical applications.

A choice of a representation $\pi(\cdot)$ corresponds to a choice of a model with observables $A_k = \pi(a_k)$ ($k = 1, \dots, n$), which are connected by the relations

$$P_k(A_1, \dots, A_n) = 0 \qquad (k = 1, \dots, m). \tag{2}$$

Since the collection of self-adjoint (bounded or unbounded) operators $(A_k)_{k=1}^n$, satisfying the relations (2), defines the representation $\pi(\cdot)$, from now on we speak of representations of the relations (2).

The notions of a irreducible (indecomposable) $*$-representation, factor representation, unitarily equivalent representations, a $*$-algebra of type I (not of type I), etc., have the same sense as one accepted in the representation theory (see, e.g., [13, 32, 25]). We only

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note that for $*$-representations the concepts of irreducibility and indecomposability mean the same thing.

2. This article contains (Sect. 1) a brief survey of papers of the Kiev mathematicians and related papers on the structure of families of bounded and unbounded operators which satisfy such relations that their study (description of their representations) can be done by studying the corresponding dynamical system (d.s.). The method of solving operator problems by using d.s. ascends to the classical papers [18, 19] (see, e.g., [12] and the bibliography therein). The new aspects in [3, 36, 23, 24, 25, 28], etc., are related to

1) a transition from representations of $*$-algebra to representations of C^*-algebra, or from representations by unbounded operators to representations by bounded operators;

2) a use of topological properties of dynamical systems, which are, in general, not one-to-one, to solve operator problems;

3) a need to consider isometries and partial isometries in the operator part of the problem.

The main object of this paper (Sect. 2–5) is a collection of examples of describing the representations of certain $*$-algebras by using d.s. In Sect. 2, examples of operator relations are generated by a non-selfadjoint operator satisfying a relation which can be reduced to the form $AB = BF(A)$ ($A \geq 0$, U being a partial isometry) which, in its turn, can be done by using the corresponding one-dimensional dynamical system. In Sect. 3, we consider examples of relations similar to ones comsidered in Sect. 2, but connecting several operators. Their study can be reduced to a study of collections of commuting self-adjoint operators $\mathbf{A} = (A_k)$, which are connected with a non-selfadjoint operator B by the relations $A_kB = BF_k(\mathbf{A})$, and depends on properties of of the corresponding multi-dimensional d.s. In Sect. 4, we consider similar relations involving a collection of commuting operators (B_l).

All these objects are taken from papers on mathematical physics. Some results on their representations are obtained by the authors, another ones are known (see references below), but can be obtained by using the d.s. formalism.

1 Representations of relations and dynamical systems

Consider the operator relation

$$A_kB = BF_k(\mathbf{A}). \tag{3}$$

Here $\mathbf{A} = (A_k)_{k=1}^n$ is a family of selfadjoint, generally speaking, unbounded commuting operators, B is a bounded (or closed unbounded) operator, $F(\cdot)$ is a continuous real function on $\mathbf{R}^n$.

1. **Unbounded operators** [3, 24]. To make sense out of relations (3), consider the polar decomposition of the operator $B = |B|U$ and the projection P into the initial space of the partial isometry U.

Definition 1 . *We say that operators* (A_k) *and* B *satisfy the relations* (3)*, if the following relations for the operators* A_k *and* B *hold*

$$E_{\mathbf{A}}(\Delta)U = UE_{\mathbf{A}}(\mathbf{F}^{-1}(\Delta)), \quad [E_{|B|}(\Delta), E_{\mathbf{A}}(\Delta')] = 0, \tag{4}$$
$$\Delta \in \mathcal{B}(\mathbf{R}^1), \quad \Delta' \in \mathcal{B}(\mathbf{R}^n),$$

where $\mathbf{F}(\cdot) = (F_1(\cdot), \dots, F_n(\cdot))\colon \mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{F}^{-1}(\Delta)$ *is a pre-image of* Δ *and* $E_{\mathbf{A}}(\cdot)$ *is a joint resolution of the identity for the commuting family* $\mathbf{A}$.

Relation (4) contains only bounded operators and causes no ambiguity. If the operators are bounded, the relations (3) are equivalent to (4).

2. **Dynamical systems** (see [36, 24], etc.). From the relations (4) it is easy to see that to the relations (3) there corresponds a dynamical system on the joint spectrum of the family $\mathbf{A}$ generated by the map $\lambda \mapsto \mathbf{F}(\lambda)$. In fact, if e_λ is a joint eigenvector of the family $\mathbf{A}$ corresponding to a joint eigenvalue λ, then Be_λ is also a joint eigenvector but corresponding to a joint eigenvalue $F(\lambda)$.

Recall that the orbit of the dynamical system is a set $\Omega_\lambda = \{\mathbf{F}^{\circ n}(\lambda) \mid n \in \mathbf{Z}\}$. A periodic orbit $\lambda, \mathbf{F}(\lambda), \dots, \mathbf{F}^{\circ n}(\lambda) = \lambda$ is called a cycle. We say that a one-to-one dynamical system on $\sigma(\mathbf{A})$ is simple, if there exists its measurable section being the set intersecting each orbit in a single point. A measure is ergodic with respect to the transform $\mathbf{F}(\cdot)$, if any $\mathbf{F}$-invariant measurable set is of zero or full measure. The condition of being simple means that any ergodic measure is concentrated on a single orbit.

It follows from (4) that for any $\mathbf{F}$-invariant set Δ, the operator $E_{\mathbf{A}}(\Delta)$ is a projection on an invariant subspace. Thus, for irreducible representation, the joint spectral measure of the family $\mathbf{A}$ is ergodic with respect to $\mathbf{F}(\cdot)$. If the dynamical system is simple, then in the irreducible case the spectral measure of the family $\mathbf{A}$ is concentrated on an orbit. Thus, there is a correspondence between irreducible representations and orbits of the dynamical system. If there is no measurable section, then, following J. von Neumann [38], one can construct a factor representation of the relations (3) which is not of the type I. In this case, there exist ergodic quasi-invariant measures which are not concentrated on a single orbit. Thus, there arise a wide class of representations which do not correspond to orbits, and the description of which is problematic.

3. **Involution conditions**. If no conditions are assumed on B, then it is a very complex task to describe all irreducible representations of (3) up to unitary equivalence. This problem contains in itself a standard wild $*$-problem: to describe up to unitary equivalence arbitrary pairs of self-adjoint operators, which in turn contain in itself a problem of description of any finite collections of self-adjoint operators (see [16, 31, 29]).

In what follows, we assume that there are additional relations between B and B^*. It follows immediately from (3) (or its accurate version (4)) that the operator $|B|$ commutes with the operators (A_k). We additionally assume that the operators U and B satisfy the relation of the form

$$|B|U = UF_{n+1}(A_1, \dots, A_n, |B|)$$

(note that this is essentially a relation between B and B^*). In particular, such a relation holds for the self-adjoint, unitary or normal operator B. Thus, the problem is reduced to the one for the relations of the form

$$A_k U = UF_k(\mathbf{A}), \tag{5}$$

which connect the collection $\mathbf{A} = (A_k)$ and the operator U, where U is a unitary, isometric or partially isometric operator. Note that the assumed conditions imply the following relations for the operator U: operators $U^l(U^*)^l$, $(U^*)^l U^l$, $p = 1, 2, \dots$ form a commuting family (operator U is centered, see, e.g., [6]).

4. **Description of representations correspoonding to an orbit** [36, 35, 28]. Fix an orbit Ω of the d.s. $\lambda \mapsto \mathbf{F}(\lambda)$. Here we study the class of irreducible representations of (5) corresponding to the orbit.

a) Let the operator U be unitary. Then the joint spectrum of the commuting family $\mathbf{A}$ is simple and the operators act on $l_2(\Omega)$ by the formulas

$$\begin{aligned} A_k e_{\mathbf{x}} &= x_k e_{\mathbf{x}}, \qquad \mathbf{x} = (x_1, \dots, x_n) \in \Omega, \\ U e_{\mathbf{x}} &= u(\mathbf{x}) e_{\mathbf{F}(\mathbf{x})}. \end{aligned} \tag{6}$$

If the dynamical system acts freely (without cycles), one can pass to a unitary equivalent representation to get $u(\mathbf{x}) \equiv 1$; in the case of cycle, one can set $u(\mathbf{x}) \equiv 1$ for all $\mathbf{x}$ but one. Thus, in the case of free action there is a unique representation corresponding to the orbit, and in the case of a cycle (including the case of stationary point, which is a 1-cycle) there exist a family of finite-dimensional irreducible representations parametrized by the points of S^1.

b) If the operator U is an isometry (or co-isometry, i.e., adjoint to isometry), the joint spectrum of the family $\mathbf{A}$ is simple, in general, only for the free action of the d.s. Under this assumption, to the orbit there corresponds a countable collection of irreducible representations which are parametrized by the point $\mathbf{x}_0 \in \Omega$ (the highest or the lowest weight). The spectral measure in this case does not fill the whole orbit but only its part $\Omega_{\mathbf{x}} = \{\mathbf{F}^{\circ l}(\mathbf{x}) \mid l \geq 0\}$. Representations act on $l_2(\Omega_{\mathbf{x}})$ by the formulas

$$\begin{aligned} A_k e_{\mathbf{x}} &= x_k e_{\mathbf{x}}, \\ U e_{\mathbf{x}} &= e_{\mathbf{F}(\mathbf{x})}. \end{aligned} \tag{7}$$

c) If the operator U is a partial isometry (but not isometry or co-isometry), and the d.s. acts freely, to the orbit, there corresponds a countable family of finite-dimensional irreducible representations of the relations (5), which are parametrized by the points $\mathbf{x}_1$, $\mathbf{x}_2 \in \Omega$ (the highest and the lowest weights). They act on $l_2(\Omega_{\mathbf{x}_1,\mathbf{x}_2})$, where

$$\Omega_{\mathbf{x}_1,\mathbf{x}_2} = \{\mathbf{x}_1, \mathbf{F}(\mathbf{x}_1), \dots, \mathbf{F}^{\circ m}(\mathbf{x}_1) = \mathbf{x}_2\},$$

by the formulas

$$\begin{aligned} A_k e_{\mathbf{x}} &= x_k e_{\mathbf{x}}, \\ U e_{\mathbf{x}} &= e_{\mathbf{F}(\mathbf{x})}, \qquad \mathbf{x} \neq \mathbf{x}_2, \\ U e_{\mathbf{x}_2} &= 0. \end{aligned} \tag{8}$$

We can generalize the relation (5), considering a commuting family (U_l) of (partial) isometries, which together with the family (A_k) satisfy the relations $A_k U_l = U_l F_k(\mathbf{A})$. Representations of such relations can be investigated by using orbits of the corresponding $\mathbf{Z}^n$-d.s. (see Sect. 4).

2 F-normal operators

Let X be a closed densely defined operator which, together with its adjoint X^*, satisfies the relation

$$X^* X = F(X X^*), \tag{9}$$

where $F(\cdot)\colon \mathbf{R} \to \mathbf{R}$ is a one-to-one measurable map. Then (see [35, 8]) by applying the polar decomposition to the operator $X = \sqrt{XX^*}\,U$ (assuming that $\ker U = \ker X$), we can reduce the study of pairs X, X^* that satisfy (9) to the unitary classification problem for pairs $C = \sqrt{XX^*}$, U such that $C \geq 0$ is self-adjoint, U is a partial isometry, and

$$CU = UF(C).$$

Thus, we can apply the formulas from Section 1 to study the irreducible representations of the relation (9). However, since $\ker U = \ker X$, to the representation with $\ker U \neq 0$ there corresponds the orbit which contains. Similarly, if $\ker U^* \neq 0$, then $\ker X^* \neq 0$ and thus, $\ker F(C) \neq 0$, i.e., the orbit again contains zero. Moreover, since $C \geq 0$, its spectrum lies in $[0, \infty)$ which means that for any $\lambda \in \sigma(C) \subset \Omega$ either $\lambda = 0$ or $\lambda > 0$, $F(\lambda) \geq 0$, and $F^{-1}(\lambda) \geq 0$. These conditions reduce the set of orbits to which correspond irreducible representations.

Example 1 (Hermitian q-plane, see [23] etc.) As an example of the relation (9), consider the pairs of operators X and X^* satisfying the relation of a (complex) q-plane

$$XX^* = qX^*X, \qquad q \in \mathbf{R}^1. \tag{10}$$

For a polar decomposition of the operator $X = UC$ we have $UC^2 = qC^2U$. Then the spectrum of the operator C^2 is invariant with respect to the transformation $\lambda \mapsto q\lambda$ ($q \neq \pm 1$) which is impossible for a bounded nonzero operator.

All irreducible integrable pairs satisfying (10) can be described as follows. Let $H_0 = \ker C$. Then (10) implies that H_0 is invariant with respect to U, C. In H_0 the operators are trivial. Thus, for the nontrivial irreducible pairs we have $\ker C = \{0\}$.

Since the operator U is unitary, it follows from (10) that the spectrum, $\sigma(C^2)$, is invariant under multiplying by q. By positivity of C^2, we also have that for $q < 0$ there are no nontrivial pairs satisfying (10). In what follows, we suppose $q > 1$. In the irreducible case, the spectrum of C^2 is concentrated on an orbit, $\sigma(C^2) = \{q^k\lambda \mid k \in \mathbf{Z}\}$, $\lambda \in [1, q)$ being an orbit parameter. Also, the operator U is unitarily equivalent to a shift operator and we have:

Proposition 1 *Any irreducible nontrivial pair satisfying* (10) *is defined by*

$$Xe_k = \sqrt{\lambda q^k}\, e_{k+1}, \qquad k \in \mathbf{Z},$$

for some $\lambda \in [1, q)$.

Remark 1 Note that the operators $X + X^*$ and $i^{-1}(X - X^*)$ are not essential selfadjoint on the span of vectors (e_k), $k \in \mathbf{Z}$ (see [2, 1]).

In this relation we used the simplest linear dynamical system $\lambda \mapsto q\lambda$. Another one-dimensional linear mapping $\lambda \mapsto q\lambda + b$ leads to so-called q-canonical commutation relations.

Example 2 (q-CCR, [17, 4, 15, 7, 23, 5], etc.) Consider the relation

$$XX^* - qX^*X = (q + 1)I. \tag{11}$$

Applying the arguments above, we put the list of irreducible representations of (11).

Proposition 2 *Let $q \in (0,1)$. The irreducible representations of* (11) *are:*
a) one-dimensional family:

$$X_1, \phi = e^{i\phi}\sqrt{\frac{1+q}{1-q}}, \qquad \phi \in [0, 2\pi);$$

b) single infinite-dimensional (*Fock representation*) : $H = l_2$,

$$X_{\infty,0}e_k = \sqrt{F^{\circ k}(0)}\, e_{k+1}, \qquad k = 0, 1, \dots,$$

$(F(\lambda) = q\lambda + (q+1),\ F^{\circ k}(\cdot)$ *is the k-th iteration of $F(\cdot)$*);
c) infinite-dimensional family: $H = l_2(\mathbf{Z})$,

$$X_{\infty,\lambda}e_k = \sqrt{F^{\circ(k+1)}(\lambda)}\, e_{k+1}, \qquad k \in \mathbf{Z},$$

the parameter $\lambda \in (F(\lambda_0), \lambda_0]$; $\lambda_0 > \frac{1+q}{1-q}$ is fixed.

If $q < -1$, there are no representations. If $q \in (-1, 0]$, there is a family of one-dimensional representations and a unique infinite-dimensional one of the form (b). For $q > 1$, only the Fock representation (b) exists.

Note that for $q < 1$ the Fock pair is bounded. On the other hand (see [2, 1, 5], for $q > 1$, the operators $P = X + X^*$ and $Q = i(X - X^*)$, X being the operator of the Fock representation, are not essential self-adjoint on the span of (e_k), $k = 0, 1, \dots$.

Example 3 (Second-degree mapping, see [36]). The study of $*$-representations of an algebra with two self-adjoint generators, satisfying the polynomial relation

$$A^2 + B^2 + \frac{1}{i}[A,B] = (A^2 + B^2 - \frac{1}{i}[A,B] - \alpha I)^2, \quad (\alpha \in \mathbf{R}^1),$$

leads to the relation $CU = UP(C)$, where C is positive, U is unitary and $P(\cdot)$ is a polynomial of the second order. By using a real linear change of coordinates, one can reduce it to the form

$$CU = U(C - \alpha I)^2), \quad (\alpha \in \mathbf{R}^1).$$

(a) For $\alpha < -1/4$ there are no representations.

(b) $\alpha = -1/4$. The spectrum of the operator $\sigma(C) \subset [1/4, \infty)$. The mapping $P(\lambda) = (\lambda + 1/4)^2$ is one-to-one on $[1/4, \infty)$. Denote by $P^{\circ k}(\cdot)$ the k-th iteration of $P(\cdot)$. The irreducible representations could be:

(1) one-dimensional: $H = \mathbf{C}^1$, $C = 1/4$, $U = e^{i\mu}$ $(\mu \in [0, 2\pi))$;

(2) infinite-dimensional: $H = l_2(\mathbf{Z})$, the operators are

$$\begin{aligned} Ae_k &= P^{\circ k}(\lambda)e_k \qquad (\lambda \in [1, (1 + 1/4)^2), \\ Ue_k &= e_{k+1} \qquad (k \in \mathbf{Z}) \end{aligned}$$

(here C is unbounded);

(c) $-1/4 < \alpha \le 0$. The spectrum of the operator $\sigma(C) \subset [x_0, \infty)$ (we set $x_{0,1} = \frac{1}{2}(2\alpha + 1 \pm \sqrt{4\alpha + 1})$). The mapping $P(\lambda) = (\lambda - \alpha)^2$ is bijective on $[x_0, \infty)$. The irreducible representations are:

(1) one-dimensional: $H = \mathbf{C}^1$, $C = x_0$, $U = e^{i\mu}$ ($\mu \in [0, 2\pi)$) and $C = x_1$, $U = e^{i\mu}$ ($\mu \in [0, 2\pi)$);

(2) infinite-dimensional with bounded C: $H = l_2(\mathbf{Z})$,

$$\begin{aligned} Ce_k &= P^{\circ k}(\lambda)e_k, \\ Ue_k &= e_{k+1} \qquad (\lambda \in ((\lambda_0 - \alpha)^2, \lambda_0],\ k \in \mathbf{Z}) \end{aligned}$$

(here $\lambda_0 \in (x_0, x_1)$ is fixed), and with unbounded C:

$$\begin{aligned} Ce_k &= P^{\circ k}(\lambda)e_k, \\ Ue_k &= e_{k+1} \qquad (\lambda \in [\lambda_1, (\lambda_1 - \alpha)^2),\ k \in \mathbf{Z}) \end{aligned}$$

(here $\lambda_1 > x_1$ is fixed).

For $1/4 \le \alpha \le 0$, any representation of the relation can be represented as an integral of irreducible representations.

d) Consider the relation for $\alpha = \alpha^*$ ($\alpha^* = 1.4\ldots$ is a certain number such that the mapping $P(\lambda) = (\lambda - \alpha^*)^2\colon [0, x_1] \to [0, x_1]$ has cycles of all the periods equal to 2^k, $k = 1, 2\ldots$ and does not have cycles of other periods). The spectrum of the operator $\sigma(C) \subset [0, \infty)$. Then

(1) The mapping $P(\cdot)$ is bijective on $[x_1, \infty)$ and so $H'_\infty = E_C((x_1, \infty))H$ is a subspace invariant with respect to C and U. The operators C and U restricted to H'_∞ have a very simple structure, they are "glued" from irreducible ones: unbounded $Ce_k = P^{\circ k}(\lambda)e_k$ and the unitary shift operator $Ue_k = e_{k+1}$ are both defined on $l_2(\mathbf{Z})$ ($\lambda \in (x_1, P(x_1))$, $k \in \mathbf{Z}$).

(2) The structure of representations with bounded operators on $H \ominus H'_\infty$ is more complicated. The mapping $P(\cdot)\colon [0, x_1] \to [0, x_1]$ is not bijective, however, $P(\cdot)\colon K \to K$, where $K = \overline{\{P^{\circ n}(\alpha^*) \mid n = 0, 1, \ldots\}}$ is homeomorphic to the Cantor set, is one-to-one (see, e.g. [33]). The dynamical system $(K, P(\cdot))$ has a unique ergodic invariant probability measure μ_0 [20]. Following [38], we can use the measure $\mu_0(\cdot)$ to construct a factor representation of the type II_1. This shows that the problem of describing an infinite-dimensional representation of the relation with bounded operators is wild for $\alpha = \alpha^*$.

e) For $\alpha > \alpha^*$ the corresponding d.s. and, consequently, the relation have representations which generate a factor not of the type I.

Example 4 (Algebra of polynomials on a two parameter quantum disc, [14]). For $0 \le q \le 1$ and $0 \le \mu \le 1$ ($(q, \mu) \ne (1, 0)$), denote by $\mathrm{Pol}_{q,\mu}(D)$ the unital $*$-algebra over $\mathbf{C}$ generated by two elements z, z^* that satisfy the following relation

$$q^{-1}(1 - z^*z) - q(1 - zz^*) = \mu(1 - zz^*)(1 - z^*z). \tag{12}$$

This algebra is called an algebra of polynomials on the two parameter quantum disc. By rewriting (12) in the form

$$z^*z = (1 + \frac{q}{\mu}) - \frac{1}{\mu^2}\frac{1}{(q^{-1}\mu^{-1} - 1) + zz^*} = F(zz^*)$$

and applying the formalism above, one can easily get a list of irreducible representations of $\mathrm{Pol}_{q,\mu}(D)$. For instance, if $0 \le \mu \le (q^{-1} - q)$, any irreducible representation is unitarily equivalent to

a) a representation of the one-dimensional series, $H = \mathbf{C}$,

$$z = e^{i\phi}, \qquad z^* = e^{-i\phi}, \qquad \phi \in [0, 2\pi);$$

b) the highest weight infinite-dimensional representation (the Fock representation), $H = l_2$, given by

$$\begin{aligned} ze_n &= \left(1 - \frac{q^{2n}(q - q^{-1})}{(q - q^{-1}) + (1 - q^{2n})\mu}\right)^{1/2} e_{n+1}, \\ z^* e_n &= \left(1 - \frac{q^{2(n-1)}(q - q^{-1})}{(q - q^{-1} + (1 - q^{2(n-1)})\mu}\right)^{1/2} e_{n-1}. \end{aligned}$$

In other cases similar formulas can be also easily obtained from the corresponding orbits.

3 Representations of relations and multi-dimensional dynamical systems

The class of relations considered in the previous section can be extended by including a family $\mathbf{A}$ of commuting self-adjoint operators which are connected with X by the relations of the form

$$\begin{aligned} A_k X &= X F_k(\mathbf{A}), \qquad k = 1, \ldots, n, \\ X^* X &= F_{n+1}(\mathbf{A}, XX^*). \end{aligned} \tag{13}$$

If we consider a polar decomposition for the operators $X = CU$, then we obtain

$$\begin{aligned} A_k U &= U F_k(\mathbf{A}), \qquad k = 1, \ldots, n, \\ CU &= U F_{n+1}(\mathbf{A}, C). \end{aligned}$$

These relations have the form (3) and their study can be carried out by using the multi-dimensional dynamical system

$$\begin{aligned} &\mathbf{F}(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) = \\ &\quad = (F_1(\lambda_1, \ldots, \lambda_n), \ldots, F_n(\lambda_1, \ldots, \lambda_n), F_{n+1}(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}). \end{aligned} \tag{14}$$

In particular, if the dynamical system (14) has a measurable section, then the joint spectrum of the commuting family $(\mathbf{A}, C)$ is concentrated on a single orbit, and the description of the representations is reduced to the calculation of the orbits and the corresponding coefficients in (6)–(8). In fact, likewise in the previous section, the conditions $C \geq 0$ and $\ker C = \ker U$ imply the corresponding restrictions on $\operatorname{supp} C$.

Example 5 (A class of quadratic algebras with three generators, see [22]). Consider an algebra with generators X, Y, Z and the relations

$$\begin{aligned} xy - qyx &= \lambda y, \\ zx - qxz &= \lambda z, \\ \alpha yz - \beta zy &= P(x), \end{aligned} \tag{15}$$

where $q \in \mathbf{R}$, $q \neq \pm 1$, $\alpha, \beta \in \mathbf{C}$, $\lambda \in \mathbf{R}$ and $P(\cdot)$ is a quadratic polynomial in x. The involution $x^* = x$, $y^* = z$ preserves the relations (16). Using $x + (1-q)^{-1}$ instead of x, we get the relations of the form

$$\begin{aligned} xy - qyx &= 0, \\ zx - qxz &= 0, \\ \alpha yz - \beta zy &= P(x), \end{aligned} \tag{16}$$

which we shall consider to the end of this section.

In the self-adjoint generators $A = x$, $B = \frac{1}{2}(y+z)$, $C = \frac{1}{2i}(y-z)$, the relations have the form

$$\begin{gathered} (1+q)[A,B] = -i(1-q)\{A,C\}, \\ (1+q)[A,C] = i(1-q)\{A,B\}, \\ \tfrac{1}{i}(\alpha-\beta)[B,C] + (\alpha+\beta)(B^2 - C^2) = \tfrac{1}{2}P(A). \end{gathered}$$

The representations of this algebra were studied in [22]. They are described by the pairs of (in general, unbounded) operators $X = X^*$, Y satisfying the relations

$$\begin{gathered} XY = qYX, \\ \alpha YY^* = P(X) - \beta Y^*Y, \end{gathered} \tag{17}$$

which are a partial case of the relations (13).

In all the cases, an irreducible representation acts on a certain orthonormal basis e_k by the formulas

$$\begin{aligned} Xe_k &= \lambda q^k e_k, \\ Ye_k &= y_k e_{k+1}. \end{aligned} \tag{18}$$

Here the range of changing k, the parameter λ and the values of $y_k \geq 0$ depend on the specific form of the second relation in (17) and on the representation. Indeed, the second relation in (17) implies

$$\alpha |y_k|^2 + \beta |y_{k+1}|^2 = P(\lambda q^k). \tag{19}$$

Fix the parameter λ. The non-negativity of y_k implies that one of the following must hold:

(i) $y_k > 0$ for all $k \in \mathbf{Z}$; in this case in (18) $k \in \mathbf{Z}$ and the representation depends on the parameter y_0;

(ii) $y_l = 0$ for some l and $y_k > 0$ for all $k > l$ or for all $k < l$; in this case in (18) $k > l$ ($k \leq l$, respectively) and the representation is uniquely determined by λ and l;

(iii) $y_l = 0$, $y_m = 0$ and $y_k > 0$ for all $k = l+1, \ldots, m-1$; in this case in (18) $k = l+1, \ldots, m$ (the representation is finite-dimensional) and the representation is determined by λ, l.

Any collection (y_k), λ, satisfying (19) and one of the conditions (i)–(iii) determines an irreducible representation of (16) and vice versa.

Example 6 (Witten's deformations of $su(2)$ and $su(1,1)$). In [9] the representations of the algebras $\mathcal{A}^{\pm}$ were studied. These algebras are generated by the elements J_0, J_+, J_-

($J_0^* = J_0$, $J_+^* = J_-$) satisfying the relations

$$\begin{aligned} [J_0, J_+] &= (1 + (1-q)J_0)J_+, \\ [J_0, J_-] &= -J_-(1 + (1-q)J_0), \\ [J_+, J_-] &= \pm 2J_0(1 + (1-q)J_0). \end{aligned}$$

These relations can be rewritten in the form

$$\begin{aligned} J_0 J_+ q^{-1} - J_+ J_0 &= q^{-1} J_+, \\ J_0 J_1 - q J_- J_0 &= -q J_-, \\ J_+ J_- - J_- J_+ &= \pm 2J_0(1 + (1-q)J_0), \end{aligned}$$

which are a partial case of the relations (15). Using the described formalism, it is easy to obtain the list of irreducible representations of these algebras (which essentially coincides with the one obtained in [9]) by bounded and unbounded operators. Note that the one-dimensional representations of these algebras form a one-parameter family (in [9] a unique one is mentioned).

This method can be also applied to a description of representations of the Sklyanin algebras (in the degenerated case) and $su_q(2)$, $su_q(1,1)$ [34] and other quantum algebras.

4 Representations of relations and d.s. generated by $\mathbf{Z}^n$

Here we consider a family of involutive algebras having generators $(b_j)_{j=1}^n$ which satisfy the relations

$$\begin{aligned} b_j b_k &= \lambda_{jk}\, b_k b_j, \\ b_j^* b_k &= \mu_{jk}\, b_k b_j^*, \\ b_j^* b_j &= F_j(b_1 b_1^*, \dots, b_n b_n^*) \end{aligned} \tag{20}$$

(λ_{jk}, $\mu_{jk} > 0$, $1 \le j, k \le n$).

Note that by the two first relations from (20), the elements $b_j^* b_j$, $j = 1, \dots, n$ form a commuting family. Indeed, for all $1 \le j, \quad k \le n$

$$b_j^* b_j\, b_k^* b_k = \mu_{kj}^{-1} b_j^* b_k^* b_j b_k = \lambda_{jk} \mu_{kj}^{-1} b_j^* b_k^* b_k b_j = \mu_{kj}^{-1} b_k^* b_j^* b_k b_j = b_k^* b_k\, b_j^* b_j.$$

This enables one to avoid problems in defining the functions $F_j(\cdot)$ of a family of (a priori non-commuting) variables.

Our further aim is to describe, up to unitary equivalence, collections of (in general setting, closed unbounded) operators B_j, $j = 1, \dots, n$ which satisfy (20). Let $B_j = C_j U_j$, $j = 1, \dots, n$ ($C_j^* = C_j$) be the polar decompositions of the (closed) operators B_j.

Lemma 1 *Let the operators B_j, $j = 1, \dots, n$ be bounded. Then for the operators $B_j = C_j U_j$, the relations* (20) *are equivalent to the following relations*

$$\begin{aligned} C_j^2 U_k = q_{jk} U_k C_j^2, &\qquad j \ne k, \\ C_j^2 U_j = U_j F_j(C_1^2, \dots, C_n^2), &\qquad j = 1, \dots n, \\ U_j U_k = U_k U_j, \qquad U_j U_k^* = U_k^* U_j, &\qquad j < k, \end{aligned} \tag{21}$$

where

$$q_{jk} = \begin{cases} \mu_{jk}\lambda_{jk}, & j < k \\ \mu_{jk}\lambda_{jk}^{-1}, & j > k \end{cases}.$$

For these operators $[(U_l^*)^i U_l^i, (U_l^*)^j U_l^j] = 0$ *and* $[(U_l^*)^i U_l^i, U_l^j (U_l^*)^j] = 0$ *for all* $i, j \geq 0$, $1 \leq l \leq n$.

PROOF. To prove the lemma, it is sufficient to substitute the polar decompositions into (20) taking into account that $U_l U_l^* = \operatorname{sign} C_l$. ■

We take the relations (21) (together with a commutativity of C_k) as a precise operator version of the relations (20) for unbounded operators.

On the space $\mathbf{R}^n$ we consider a dynamical system generated by the mappings

$$\mathbf{F}_l(x_1, \dots, x_n) = (q_{1l}x_1, \dots, q_{l-1\,l}x_{l-1}, F_l(x_1, \dots, x_n), q_{l+1\,l}x_{l+1}, \dots, q_{nl}x_n). \tag{22}$$

We consider only such relations for which

$$\mathbf{F}_j(\mathbf{F}_k(\cdot)) = \mathbf{F}_k(\mathbf{F}_j(\cdot)), \qquad j \neq k,$$

which is equivalent to the following conditions for the functions $F_j(\cdot)$:

$$F_j(\mathbf{F}_k(x_1, \dots, x_n)) = q_{jk} F_j(x_1, \dots, x_n). \tag{23}$$

This assumption is satisfied for all examples considered in this paper and in many other cases. Thus, we have a dynamical system on $\mathbf{R}^n$ with an action of the group $\mathbf{Z}^m$. Note that there are examples of another sort (see, e.g., [10, 27, 11], etc.), in which a more general dynamical systems appear.

In what follows, we also consider only such relations for which there exists a measurable section of the corresponding dynamical system. By [28], for irreducible collections the spectral measure of the commuting family C_k, $k = 1, \dots, n$ is concentrated on a single orbit of the dynamical system $\mathbf{F}_k(\cdot)\colon \mathbf{R}^n \to \mathbf{R}^n$, $k = 1, \dots, n$.

For a fixed orbit Ω, we now describe all the irreducible collections $(B_k)_{k=1}^n$, satisfying (20). Let $\Delta \subset \Omega$ be the support of the spectral measure of the commuting family $(C_j^2)_{j=1}^n$. Fine points of the relations (21) are: 1) the operators C_j, $j = 1, \dots, n$ are non-negative and 2) $U_j U_j^*$ is a projection on $(\ker C_j^2)^\perp$.

Now we consider the possible types of orbits and describe the corresponding irreducible representations of (20).

Theorem 1 *Any irreducible representation acts in the space* $l_2(\Delta)$. *For each* $l = 1, \dots, n$, *one of the following is true:*

a) The mapping $\mathbf{F}_l(\cdot)$ *has a stationary point* $\mathbf{x} \in \Delta$ *(in this case all the points of* Δ *are also stationary for* $\mathbf{F}_l(\cdot)$*). If* $x_l = 0$, *then* $B_l = 0$*; otherwise* B *has the form*

$$B_l e_{\mathbf{x}} = \beta_l \, x_l \, e_{\mathbf{x}},$$

where β_l *is a parameter equal by its absolute value to one;*

b) The mapping $\mathbf{F}_l(\cdot)$ has no stationary points. Now the operator B_l has the form

$$B_l e_{\mathbf{x}} = F_l(\mathbf{x})\, e_{\mathbf{F}_l(\mathbf{x})}. \tag{24}$$

Here, the kernel of the operator B_l is a span of the vectors $e_{\mathbf{x}}$ such that $F_l(\mathbf{x}) = 0$; the kernel of the adjoint operator B_l^* is generated by the vectors $e_{\mathbf{x}}$ for which $x_l = 0$.

Remark 2 In the case b), different situations occur. Depending on whether the l-th coordinate of the point $\mathbf{x} \in \Delta$ is zero or not, the operator B_l or B_l^* has a nontrivial kernel or not.

Example 7 (The Heisenberg relations for the quantum $E(2)$ group [39, 26]).

In this section we study the $*$-representations of the involutive algebra $\mathcal{A}$ generated by the so-called Heisenberg relations [39]. These relations connect the generators of the quantum deformation of $E(2)$ group and ones of its dual.

The quantum deformation of the group of motions was introduced and investigated in [37, 40, 39]. The algebra of "functions on $E_q(2)$" is an algebra generated by the elements v and n, v being unitary and n being normal, satisfying the relation

$$vn = qnv, \qquad q > 0. \tag{25}$$

On the other hand, using the comultiplication in A, the algebra of "continuous functions on $\widehat{E}_q(2)$", where $\widehat{E}_q(2)$ denotes the Pontryagin dual of $E_q(2)$, was constructed and investigated in [39]. This algebra is generated by the elements N and b (N is self-adjoint, b is normal) with the relation

$$Nb = b(N + I). \tag{26}$$

If one consider both the algebras represented on the same Hilbert space, some natural relations between the generators v, n and the generators N, b (the Heisenberg relations, see [39]) appear. These relations are:

$$\begin{gathered} vN = (N - I)v, \qquad vb = q^{-1/2}bv, \qquad nN = (N + I)n, \\ bn^* = q^{1/2}n^*b, \qquad nb - q^{1/2}bn = (1 - q^2)q^{-\frac{N+I}{2}}v. \end{gathered} \tag{27}$$

In the sequel we consider the $*$-algebra $\mathcal{A}$ generated by the elements v, n, N and b such that v is unitary, N is self-adjoint, n and b are normal and the generators satisfy the relations (25), (26), (27).

Instead of b and N, introduce the new generators, $d = bv^*$ and $M = (1 - q^2)q^{-\frac{N+I}{2}}$. Then the relations are:

$$\begin{gathered} vn = qnv, \qquad nn^* = n^*n, \qquad Md = q^{1/2}dM, \qquad d^*d = q^{-1}dd^*, \\ vM = q^{1/2}Mv, \qquad vd = q^{-1/2}dv, \qquad nM = q^{-1/2}Mn, \\ nd^* = q^{-1/2}d^*n, \qquad nd - q^{3/2}dn = M. \end{gathered}$$

Considering representations of the algebra $\mathcal{A}$, we have to keep in mind that the operator M must be positive for $0 < q < 1$ and negative for $q > 1$. We will consider the case $0 < q < 1$ (the case $q > 1$ is quite similar).

We use the following statement.

Lemma 2 *Suppose we have a ∗-representation of the Heisenberg relations (by, generally speaking, unbounded operators) and there exists a vector $f \in H$ such that $f \in \ker n$ and*

$$nd\, f - q^{3/2} dn\, f = M\, f$$

(it is supposed that the needed operators are defined on f). Then $f = 0$.

Proof. Indeed, since n is normal, then $f \in \ker n^*$. But this implies

$$(Mf, f) = (nd\, f - q^{3/2} dn\, f, f) = (nd\, f, f) = (df, n^* f) = 0,$$

which is impossible by the positivity of M. ■

Introduce the element

$$y = nMd + \frac{q^{-1/2}}{q^2 - 1} M^2.$$

By the previous Lemma, we can suppose that for "good" representations of the Heisenberg relations, the operators n and M are invertible and that if we find y, we shall be able to reconstruct d as

$$d = M^{-1} n^{-1} y + \frac{1}{1 - q^2} n^{-1} M.$$

So, replacing d by y we get the following relations:

$$\begin{gathered} vn = qnv, \qquad nn^* = n^* n, \qquad yM = My, \qquad [y, y^*] = 0, \\ vM = q^{1/2} Mv, \qquad vy = qyv, \qquad nM = q^{-1/2} Mn, \\ n^* y = qyn^*, \qquad ny = qyn. \end{gathered} \tag{28}$$

From now on, we deal with ∗-representations of the algebra $\mathcal{A}$.

Proposition 3 *There are no representations of* (28) *by bounded operators.*

Proof. Indeed, since $Mu = q^{-1/2} uM$ and u is unitary, the spectrum of M is invariant under the multiplication by $q^{-1/2}$. But since $M > 0$, the spectrum of M does not contain zero and thus is unbounded. ■

In the space $l_2(\mathbf{Z})$, introduce the operators

$$Se_k = e_{k+1}, \qquad Te_k = ke_k, \qquad Qe_k = q^{k/2} e_k = e^{\frac{1}{2}T} e_k.$$

Using the technique developed in [24, 36, 28], one can calculate all the irreducible representations of the algebra $\mathcal{A}$.

Theorem 2 *All the irreducible ∗-representations of the algebra $\mathcal{A}$ up to unitary equivalence are:*

a) *representations in $l_2(\mathbf{Z}) \otimes l_2(\mathbf{Z})$:*

$$\begin{aligned} n &= \lambda\, S \otimes Q^2, \\ v &= S^* \otimes S^*, \\ N &= \alpha - T \otimes I, \\ b &= \frac{q^{-\alpha/2-1}}{\lambda} Q(S^*)^2 \otimes Q^{-2} S^*; \end{aligned}$$

b) *representations in* $l_2(\mathbf{Z}) \otimes l_2(\mathbf{Z}) \otimes l_2(\mathbf{Z})$*:*

$$\begin{aligned} n &= \lambda S \otimes Q^2 \otimes I, \\ v &= S^* \otimes S^* \otimes S^*, \\ N &= \alpha - T \otimes I \otimes I, \\ b &= \frac{\delta q^{(\alpha+1)/2}}{\lambda(1-q^2)} Q^{-1}(S^*)^2 \otimes Q^{-2} \otimes Q^2 S^* + \frac{q^{-\alpha/2-1}}{\lambda} Q(S^*)^2 \otimes Q^{-2} S^* \otimes S^*, \end{aligned}$$

where $\lambda, \delta \in (q, 1]$, $\alpha \in [0, 1)$.

Proof. Follows from Theorem 1 applied to (28). ■

Example 8 (Nonstandard real quantum sphere[21, 28]).

The algebra of functions on the nonstandard real quantum sphere (see [21]) is an associative $*$-algebra over $\mathbf{C}$ with the generators x, y, u, v, c, d and the relations

$$\begin{gathered} ux = qxu, \qquad vx = qxv, \qquad yu = quy, \qquad yv = qvy, \\ vu - uv = (q - q^{-1})\, d, \qquad xy - q^{-1} uv = yx - qvu = c + d, \\ dx = q^2 xd, \qquad dv = q^2 vd, \qquad ud = q^2 du, \qquad yd = q^2 dy, \end{gathered} \tag{29}$$

c being central, and the involution is given by $x^* = y$, $u^* = -q^{-1}v$, $c^* = c$, $d^* = d$. One can rewrite the relations in the form

$$\begin{gathered} ux = qxu, \qquad u^* x = qxu^*, \\ u^* u = q^{-2} uu^* - (1 - q^{-2})(xx^* - c), \qquad x^* x = q^2 xx^* + (1 - q^2)c, \\ d = xx^* + uu^* - c \end{gathered}$$

involving only x, u, c, d.

Given a unitary representation, the operators in the polar decompositions $X = C_x U_x$ and $U = C_u U_u$ satisfy the relations

$$\begin{aligned} &C_x^2 U_u = U_u C_x^2, \qquad C_u^2 U_x = q^2\, U_x C_u^2, \\ &C_x^2 U_x = U_x (q^2 C_x^2 + (1 - q^2) cI), \\ &C_u^2 U_u = U_u (q^{-2} C_u^2 - (1 - q^{-2})(C_x^2 - cI)). \end{aligned}$$

The corresponding dynamical system on $\mathbf{R}^2$ is generated by the mappings

$$\begin{aligned} \mathbf{F}_1(x_1, x_2) &= (q^2 x_1 + (1 - q^2)c, q^2 x_2), \\ \mathbf{F}_2(x_1, x_2) &= (x_1, q^{-2} x_2 - (1 - q^{-2})(x_1 - c)), \end{aligned}$$

which are easily checked to satisfy the conditions (23).

An orbit of the dynamical system consists of the points

$$\mathbf{x}^{(kl)} = \mathbf{F}_1^k(\mathbf{F}_2^l(\mathbf{x})) = (q^{2k} x_1 + (1 - q^{2k})\, c, q^{2(k-l)} x_2 - q^{2k}(1 - q^{-2l})(c - x_1)),$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{F}_l^k(\cdot)$ is the k-th iteration of the mapping $\mathbf{F}_l(\cdot)$.

The mapping $\mathbf{F}_1(\cdot)$ has the unique stationary point $(c, 0)$; the stationary points of the mapping $\mathbf{F}_2(\cdot)$ are $(x, c-x)$. No orbit has cycles.

Below, we present a list of the orbits of the corresponding sets Δ and the corresponding irreducible representations.

1) *The stationary point* $(c, 0)$. For $c = 0$ there is the trivial representation $X = U = 0$, and for $c > 0$ to the orbit there corresponds a family of one-dimensional representations $U = 0$, $X = \alpha\, c$, where $|\alpha| = 1$. Thus, the set of one-dimensional irreducible representations is parametrized by the points of a circle.

2) Among the orbits which are *invariant with respect to* $\mathbf{F}_2(\cdot)$, representations correspond only to the orbit passing through the point $\mathbf{x} = (0, c)$. The set Δ consists of the points $\mathbf{x}^{(k)} = ((1-q^{2k})c, q^{2k}c)$, and the irreducible representations corresponding to the orbit act on l_2 by the formulas

$$\begin{aligned} Xe_k &= \sqrt{(1-q^{2k})\,c}\,e_{k+1},\\ Ue_k &= \alpha\, q^{k-1}\sqrt{c}\,e_k, \qquad |\alpha| = 1,\ k = 1, 2, \dots \end{aligned}$$

The parameters for this family of representations are $c > 0$, and $\alpha \in S^1$.

3) For $c > 0$, the *orbits passing through* (c, y), $y > 0$ are contained completely in the first quadrant. They consist of the points (c, q^{2n}), $n \in \mathbf{Z}$; their set is numbered by the points of the circle S^1. The corresponding irreducible representations act in $l_2(\mathbf{Z})$ by the formulas

$$\begin{aligned} Xe_k &= \sqrt{c}e_{k+1},\\ Ue_k &= \lambda q^k\, e_{k-1}. \end{aligned}$$

The parameters for this family of representations are $\lambda \in (q^2, 1] \approx S^1$ (orbit parameter) and $c \geq 0$.

4) For $c > 0$, there exist representations corresponding to *the orbits passing through the points* $(0, y)$, $y > c$. Such orbits are also numbered by the points of the circle. The set Δ consists here of the points

$$\mathbf{x}^{(k,l)} = ((1-q^{2k})\,c, q^{2(k-l)}\lambda + q^{2k}(1-q^{-2l})\,c), \qquad k \geq 0,\ l \in \mathbf{Z}.$$

The corresponding irreducible representations in $l_2(\mathbf{N}\times\mathbf{Z})$ are given by the formulas

$$\begin{aligned} Xe_{kl} &= \sqrt{(1-q^{2k})\,c}\,e_{k+1,l},\\ Ue_{kl} &= \sqrt{q^{2(k-l-1)}\lambda + q^{2k-2}(1-q^{-2l})\,c}\,e_{k,l+1}, \qquad k = 1, 2, \dots;\ l \in \mathbf{Z}. \end{aligned}$$

The parameters for this family are $\lambda \in (c+q^2, c+1]$ (orbit parameter) and $c \geq 0$.

5) For $c > 0$, there exists one more family of representations depending on continuous parameter. These are ones *corresponding to the orbits passing through the points* $(x, 0)$, $x > c$. Such orbits are numbered by the points of $\lambda \in (c+q^2, c+1] \approx S^1$. The set Δ for such orbit is $\Delta = \{\mathbf{x}^{(kl)} = (c - q^{2k}(c-\lambda), q^{2k}(1-q^{-2l})(c-\lambda)),\ l \geq 0, k \in \mathbf{Z}\}$; the irreducible representation corresponding to the parameters c, λ acts on $l_2(\mathbf{Z}\times\mathbf{N})$:

$$\begin{aligned} Xe_{kl} &= \sqrt{c - q^{2k+2}(c-\lambda)}e_{k+1,l},\\ Ue_{kl} &= q^k\sqrt{(1-q^{-2l})(c-\lambda)}e_{k,l+1},\\ &k \in \mathbf{Z}, \quad l = 1, 2, \dots \end{aligned}$$

Unlike the previous case, for $c < 0$ to the orbit passing through the origin $(0,0)$, there corresponds an irreducible representation of this family on $l_2(\mathbf{N}\times\mathbf{N})$:

$$\begin{aligned} Xe_{kl} &= \sqrt{(1-q^{-2k+2})\,c}e_{k-1,l}, \\ Ue_{kl} &= q^{-k}\sqrt{(1-q^{-2l})\,c}\,e_{k,l+1}, \qquad k,l=1,2,\ldots \end{aligned}$$

6) Finally, for $c > 0$ to the orbit passing through the origin $(0,0)$, there corresponds an irreducible representation (the Fock representation). The set Δ is now $\Delta = \{\mathbf{x}^{(kl)} = ((1-q^{2k})\,c, q^{2k}(1-q^{-2l})\,c),\ k\ge 0, l\le -1\}$, and the representation acts on $l_2(\mathbf{N}\times\mathbf{N})$ by the formulas:

$$\begin{aligned} Xe_{kl} &= \sqrt{(1-q^{2k})\,c}\,e_{k+1,l}, \\ Ue_{kl} &= q^{k-1}\sqrt{(1-q^{2l})\,c}\,e_{k,l-1}, \qquad k,l=1,\ldots \end{aligned}$$

Note that the operators of this representation, as like as in the cases 1) and 2), are bounded.

Note that the twisted CCR relations [30] also fit into the class (20). The class of unbounded representations described in [30] can be obtained by using Theorem 1.

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