Nonlinear Schrödinger Equations and the Separation Property

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Abstract

We investigate hierarchies of nonlinear Schrödinger equations for multiparticle systems satisfying the separation property, i.e., where product wave functions evolve by the separate evolution of each factor. Such a hierarchy defines a nonlinear derivation on tensor products of the single-particle wave-function space, and satisfies a certain homogeneity property characterized by two new universal physical constants. A canonical construction of hierarchies is derived that allows the introduction, at any particular "threshold" number of particles, of truly new physical effects absent in systems having fewer particles. In particular, if single quantum particles satisfy the usual (linear) Schrödinger equation, a system of two particles can evolve by means of a fairly simple nonlinear Schrödinger equation without violating the separation property. Examples of Galileian-invariant hierarchies are given.

1   Introduction

A growing literature is devoted to properties of nonlinear Schrödinger equations. These are usually introduced for one of two reasons: (a) to describe particular physical effects phenomenologically (such as plasma waves), or (b) to explore fundamental arguments that the basic equations of quantum mechanics might be, or even ought to be, nonlinear (with the usual linear theory representing only an approximation). Equations introduced on foundational grounds require special scrutiny since they could, if found to be tenable, profoundly change our views of basic physics. A wide variety of different nonlinearities, stemming from distinct philosophies and approaches, have been proposed [1–8, 16–18]; the cited references are just a small subset, that includes the articles from which we drew some of the conclusions presented here. Of course, we cannot do full justice to each proposed equation or set of equations. Clearly some sort of systematization is required, toward which we believe our results contribute.

In the present paper we elucidate and explore one desirable property of nonlinear Schrödinger equations, the so-called separation property in quantum mechanics articulated in 1976 by Bialynicki-Birula and Mycielski [1]. This property relates the time-evolutions of systems comprised different numbers of particles. For a hierarchy of N-particle time-evolution equations, it expresses the physical condition in quantum mechanics that initially uncorrelated, noninteracting particle subsystems remain uncorrelated as they evolve; i.e., that in the absence of interaction potentials, subsets of particles evolve according to their own evolution equations as if the remaining particles were not present. It is the separation property that, in principle, allows us to speak of an experiment performed on an "isolated" system of quantum particles.

We shall make the interesting observation that, for nonlinear time-evolution equations, the separation property allows some non-uniqueness in the hierarchies that are permitted. In fact, if the N-particle equations of motion in a hierarchy are specified for N less than some fixed threshold value, the separation property allows the possibility of new nonlinear terms appearing for the first time at the threshold value of N. These terms can resemble in form the nonlinear terms introduced recently by Doebner and Goldin [3]. In particular, if single quantum particles satisfy the usual, linear Schrödinger equation, a system of two particles can evolve by means of a fairly simple nonlinear Schrödinger equation without violating the separation property.

There are arguments that nonlinearity in the time-evolution has undesirable consequences for quantum mechanics. One argument is that any nonlinearity, however small, would permit controlled superluminal signals to be sent via long-range quantum correlations of the Einstein-Podolsky-Rosen (EPR) type [9, 10]. This is fundamentally incompatible with special relativity. Another argument, brought forth specifically for the Weinberg[8] theory, is that the nonlinearity requires either allowing communication between different branches of an Everett-type multiple-world quantum universe, or else allowing that physical systems react to the content of the experimenter's mind [11].

As compelling as these objections may seem, accepting that the derived consequences of the nonlinearity are undesirable, it may yet be premature to bow to them. The results mentioned all depend on maintaining in nonlinear theories the standard quantum-mechanical rules concerning state-function behavior when measurements occur. However, it is generally believed that the behavior under measurement has some relation to the time-evolution; a relation presumably obtained by considering the measuring instrument together with the observed quantum system as a larger system governed by the usual time-evolution equations with a macroscopic number of particles. This approach has its own difficulties, and the usual rules of quantum measurement are only obtained by going to the limit of an infinite apparatus [12]. Nevertheless it raises the counter argument that in changing the evolution to be nonlinear, the assumptions about what happens when measurements occur should likewise undergo modifications — possibly eliminating the difficulties mentioned. There are other arguments for linearity, for example those provided by Jordan [13]. To come to grips fully with such issues, it is necessary to explore nonlinear evolutions in more detail. A systematic study is called for even if, in the end, one concludes that such equations cannot describe fundamental physics. We believe such a conclusion is not yet warranted.

2   Characterization and Consequences of the Separation Property

Here we discuss only noninteracting systems. We assume interacting systems are to be described by adding appropriate terms, such as inter-particle or external potentials, to the equations describing noninteracting systems. With nonlinear evolutions, it is even problematic to state in advance exactly what is meant by a lack of interaction, or to identify the interaction terms in an equation. One property of interactions, however, is that they generally produce correlations in initially uncorrelated systems. The absence of this effect can be taken as expressing the lack of interaction.

In what follows we maintain the usual interpretation of the wave function ψ as a probability amplitude, so that $|\psi|^2$ is the joint probability density in configuration space for the particle positions. We disregard spin, and any internal degrees of freedom. By an "uncorrelated" system we mean that ψ is the product of two fewer-particle wave functions depending on different particle coordinates. What we shall mean by the separation property is that the evolved total wave function continues to be a product, in which the two factors evolve according their respective evolution laws as though the particles described by the other factor did not exist. This incorporates the notion of lack of interaction. Given this property, the physical laws governing a given system will be independent of the existence of other systems uncorrelated to the given one.

The separation property is thus a property not of a single equation but of a hierarchy of equations, one for each system having its own number and species of particles. Here we are disregarding considerations of particle statistics due to indistinguishability; and by not imposing any permutation symmetry properties on the wave functions, we implicitly treat all of the particles as distinguishable. However, two such particles may obey identical one-particle evolution equations, in which case we say they belong to the same species of particle.

We write the equations in a hierarchy as

$$i\hbar\partial_t\psi^{(s)} = F_s(\psi^{(s)}) \tag{1}$$

where $\psi^{(s)} = \psi^{(s)}(t, x_1, \ldots, x_N)$ is a time-dependent N-particle complex-valued wave-function, the $x_k = (x_k^1, \ldots, x_k^d)$ are the position coordinates of the k-th particle in d-dimensional space, and s is an N-tuple of particle species labels, $s = (s_1, \ldots, s_N)$, indicating to which species each one of the particles belongs. We shall drop this index from the wave-function when no confusion can arise. In (1) F_s is a quite general operator, not necessarily a linear one, depending on ψ and $\bar\psi$. We adopt a formal approach, disregarding domain considerations for all operators considered, and questions of the existence and uniqueness of solutions to the initial-value problem for the time evolution.

We shall generally use the letter ψ for time-dependent wave-functions, and ϕ for functions of just the particle positions.

It is desirable to adopt some conventions or assumptions reflecting the arbitrariness of how one labels the physical particles with the integers $\{1, \ldots, N\}$. We opt to have a multiplicity of equations describing the same physical system; that is, for any permutation π of $\{1, \ldots, N\}$, any N-particle species label s, and any function $\phi(x_1, \ldots, x_N)$, we set

$\pi s = (s_{\pi(1)}, \ldots, s_{\pi(N)})$ and $(\pi\phi)(x_1, \ldots, x_N) = \phi(x_{\pi(1)}, \ldots, x_{\pi(N)})$, and then assume that

$$F_{\pi s}(\pi\phi) = \pi(F_s(\phi)). \tag{2}$$

Thus, the operators F_s with permuted indices describe the same physical system with the particles labeled in a different manner.

Definition 1. Let F be a family of multi-particle operators indexed by species labels. Denote by $F_{[N]}$ the sub-family of N-particle operators. If the subfamily $F_{[N]}$ satisfies (2), we say it satisfies the *permutation property,* and if each such subfamily does so, we say the same of F.

We now come to the separation property. Let $n_1, \ldots, n_r$ be positive integers corresponding to the numbers of particles in each of r different subsystems. Let $N = n_1 + \cdots + n_r$ be the total number of particles. Let ϕ_j $(j = 1, \ldots, r)$ be functions whose arguments are respectively the n_j position variables. Let $\phi_1 \cdot \phi_2 \cdots \phi_r$ be the (tensor) product function of N position variables given by the product

$$\phi_1(x_1, \ldots, x_{n_1})\phi_2(x_{n_1+1}, \ldots, x_{n_1+n_2}) \cdots \phi_r(x_{n_1+\cdots+n_{r-1}+1}, \ldots, x_r). \tag{3}$$

Let $s^{(j)}$ be the n_j-tuple of species labels for the j-th subsystem, and s be the N-tuple of species labels obtained by concatenating the $s^{(j)}$ in order of increasing j. The separation property now requires that given functions ψ_j of time and n_j particle position variables that are solutions of the n_j-particle evolution equations with species labels $s^{(j)}$, then $\psi_1 \cdot \psi_2 \cdots \psi_r$ must be a solution of the N-particle evolution equation with species label s. Thus

$$i\hbar\partial_t(\psi_1 \cdots \psi_r) = F_s(\psi_1 \cdots \psi_r). \tag{4}$$

Had we not assumed the permutation property, we would have had to consider tensor products in which the order of the variables is not the same as that given in (3); but with this assumption these combinatorially more complex situations are equivalent to the ones considered.

Definition 2. We say a hierarchy of evolution equations satisfies the *separation property* if (4) holds, where the symbols are defined in the previous paragraph.

Using

$$\partial_t\psi_1 \cdots \psi_r = \partial_t\psi_1 \cdot \psi_2 \cdots \psi_r + \psi_1 \cdot \partial_t\psi_2 \cdots \psi_r + \cdots + \psi_1 \cdot \psi_2 \cdots \psi_{r-1} \cdot \partial_t\psi_r$$

together with the evolution equations for the factors, we see that the hierarchy formally satisfies the separation property if and only if

$$F_{s^{(1)}}(\phi_1) \cdot \phi_2 \cdots \phi_r + \cdots + \phi_1 \cdots \phi_{r-1} \cdot F_{s^{(r)}}(\phi_r) = F_s(\phi_1 \cdots \phi_r) \tag{5}$$

for all functions ϕ_j of n_j particle position variables. We can rewrite this in a more elegant form as:

$$\frac{F_{s^{(1)}}(\phi_1)}{\phi_1} + \cdots + \frac{F_{s^{(r)}}(\phi_r)}{\phi_r} = \frac{F_s(\phi_1 \cdots \phi_r)}{\phi_1 \cdots \phi_r}. \tag{6}$$

Thus, any F_s is determined *on product functions* by operators having lower-order indices obtained from a partition of s. Such constraints, however, do not fix F_s uniquely,

given the operators with lower-order indices. One can always add any operator that vanishes identically on product functions. We shall investigate such non-uniqueness below.

Definition 3. A family $F = (F_s)$ of multi-particle operators indexed by species labels is called a *tensor derivation* or simply a *derivation* if (6) holds.

The reason for this name is that (5) and (6) are formally the Leibniz rule for a tensor product, and have their roots in the fact that ∂_t is a derivation.

A hierarchy of evolution equations thus satisfies the separation property if the corresponding family of operators is a tensor derivation. The converse should also be true under some general conditions, but this involves proving existence and uniqueness theorems. It is useful to keep the notions separate.

Although (6) may be used to define F_s on product functions by means of the left-hand side, it is important to note that the operators on the left-hand side cannot be given freely. To see this, let $k_j \, (j = 1, \ldots, r)$ be any complex numbers with $\prod k_j = 1$. Replacing ϕ_j by $k_j \phi_j$, the right-hand side of (6) does not change; but the left-hand side becomes

$$\frac{F_{s^{(1)}}(k_1 \phi_1)}{k_1 \phi_1} + \cdots + \frac{F_{s^{(r)}}(k_r \phi_r)}{k_r \phi_r} \,. \tag{7}$$

Thus (7) must be independent of the numbers k_j. Using this for $r = 2$ and setting $s^{(1)} = a$, $s^{(2)} = b$ we have

$$\frac{F_a(k\phi_1)}{k\phi_1} - \frac{F_a(\phi_1)}{\phi_1} = \frac{F_b(\phi_2)}{\phi_2} - \frac{F_b(k^{-1}\phi_2)}{k^{-1}\phi_2} \,.$$

As each side of this equation depends on different particle position variables and different species labels, both sides must be a function $c(k)$ of k only. Thus we deduce: $F_s(k\phi) = kF_s(\phi) + kc(k)\phi$. Setting $k = 1$ in this equation, we obtain $c(1) = 0$. Using expression (7), we deduce that $\sum_j c(k_j)$ is independent of k_j. From $1 \cdot 1 = z \cdot z^{-1}$ one deduces $c(z^{-1}) = -c(z)$ and then from $1 \cdot 1 \cdot 1 = (zw)^{-1} \cdot z \cdot w$ one has $c((zw)^{-1}) + c(z) + c(w) = 0$. Thus we have the functional equation, $c(zw) = c(z) + c(w)$. Using this on the polar form $z = re^{i\theta}$, we find $c(z) = c(r) + c(e^{i\theta})$. Specializing to positive real numbers, and then to unimodular complex numbers, we have:

$$c(rs) \quad = \quad c(r) + c(s) \,,$$

$$c(e^{i(\theta+\phi)}) \quad = \quad c(e^{i\theta}) + c(e^{i\phi}) \,.$$

To solve the first of these, note that if $c(r)$ is locally integrable, we can integrate both sides with a $\mathcal{C}^\infty$ function $f(s)$ having compact support on the positive real axis. Then one has that c is a constant plus a (product) convolution of itself with f; thus c must be $\mathcal{C}^\infty$. Differentiating with respect to s and setting $s = 1$, we get $rc'(r) = c(r)$; which means that $c(r) = p \ln r$ for some complex number p.

To solve the second equation, we set $d(\theta) = c(e^{i\theta})$; then $d(\theta + \phi) = d(\theta) + d(\phi)$, and proceeding in a similar manner, we find that any locally integrable solution is of the form $d(\theta) = iq\theta$ for some complex number q (we put in the factor i for mathematical convenience). We thus find that the locally integrable solutions for c are of the form

$$c(z) = p \ln |z| + iq \arg z.$$

If $q \neq 0$, then the above solution is not everywhere continuous; but it can be chosen continuous on any complex domain where $\arg z$ can be defined in a single-valued manner. Alternatively, we can consider c to be defined on the Riemann surface of the logarithm function.

We note that $p \ln |z| + iq \arg z$ can be construed as $\ln z^{(p,q)}$ where, for $z = re^{i\theta}$, one defines the "mixed power" $z^{(p,q)} = r^p e^{iq\theta}$ as an operation in which the two factors in the polar form of z are raised to different powers! Such an operation is natural to physics, as the phase and modulus of the wave function play very different physical roles — and thus can acquire different treatments.

We have now shown the following result.

Theorem 1. *If F is any tensor derivation, then there are complex numbers p and q such that for any complex number k and species label s, we have*

$$F_s(k\phi) = kF_s(\phi) + k(p \ln |k| + iq \arg k)\phi. \qquad (8)$$

We shall not delve further here into how the ambiguity in defining the arg function is to be resolved in practice. In what follows, it is enough to assume that arg is defined as a continuous function on an open set containing the positive real axis, that it vanishes on that axis, and that $\arg k$ is sufficiently small for all possible multipliers.

Note that p and q are characteristic of the *whole hierarchy*. This means that if the nature obeys non-linear quantum mechanics with the separation property, then the logarithmic indices are new, universal physical constants with the dimension of energy. We shall see below that probability conservation implies p and iq are real. The existence of new universal physical constants in nonlinear theories was already noted by Bialynicki-Birula and Mycielski [1]. In their theory $q = 0$ (and p is real), so they spoke only of one constant.

Definition 4. The property expressed by (8) we call the *mixed-logarithmic homogeneity property*. We say of an evolution operator that satisfies this property that it is *mixed-logarithmic homogeneous*. We call the complex numbers p, q respectively the first and second *logarithmic index*. If $p = q = 0$, we say we have *strict* homogeneity, and an evolution operator whose indices are zero is said to be *strictly homogeneous*.

The failure of strict homogeneity means that if ψ is a solution of $i\hbar \partial_t \psi = F_s(\psi)$, then a multiple $k\psi$ is not a solution. However, we can multiply by a time-dependent factor $w(t)$ and require that $w(t)\psi$ be again a solution. This leads to the equation

$$i\hbar \partial_t w = (p \ln |w| + iq \arg w)w.$$

Let $w(t) = |w(t)|e^{i\theta(t)}$, $p = |p|e^{i\sigma}$, and $q = |q|e^{i\tau}$. Then the above displayed equation is equivalent to the system,

$$\partial_t \begin{pmatrix} \ln |w| \\ \theta \end{pmatrix} = \frac{1}{\hbar} \begin{pmatrix} |p| \sin \sigma & |q| \cos \tau \\ -|p| \cos \sigma & |q| \sin \tau \end{pmatrix} \begin{pmatrix} \ln |w| \\ \theta \end{pmatrix} \qquad (9)$$

which, being a linear system with constant coefficients, has a unique solution for all times once $w(0)$ is given. We can let $w(0)$ be any non-zero complex number.

It is instructive to verify the homogeneity property for various equations that have been proposed in the literature. Phenomenological equations generally do not satisfy it,

while those proposed on fundamental grounds generally do — even though the authors may not have explicitly considered the separation property. Consider the one-particle equations of the form $i\hbar\partial_t\psi = -(\hbar^2/2m)\nabla^2\psi + \kappa G(\psi)$, where κ is a (real or complex) physical constant coefficient. Typical expressions for $G(\psi)$ taken from the literature are the following:

$$(NLSE) \qquad\qquad |\psi|^2\psi$$

$$(DG) \text{ and } (FC) \qquad \nabla^2\psi + (|\nabla\psi|^2/|\psi|^2)\psi$$

$$(BM) \qquad\qquad \ln|\psi|\,\psi$$

$$(K) \qquad\qquad \ln(\psi/\bar\psi)\,\psi\,.$$

These are, respectively, the much studied phenomenological nonlinear Schrödinger equation [14], and the equations introduced by Doebner and Goldin [3], Fushchych and Cherniha [16–18], Bialynicki-Birula and Mycielski [1], and Kostin [4]. Among them (NLSE) is not mixed-logarithmic homogeneous, (DG) is strictly homogeneous, (BM) has $p \neq 0$ and $q = 0$, while (K) has $p = 0$ and $q \neq 0$.

It is interesting to note that if G is any mixed-logarithmic homogeneous operator with logarithmic indices p and q, then $G_0(\phi) = G(\phi) - [p\ln|\phi| + (q/2)\ln(\phi/\bar\phi)]\phi$ is strictly homogeneous. Thus, *modulo* strictly homogeneous operators, the (BM) and (K) terms give the most general way to deviate from strict homogeneity. This may be a bit misleading, though, since it does not mean that in a given expression one will see such terms. For instance, $G(\phi) = (\ln|\nabla\phi|)\phi$ has $p \neq 0$ but no explicit (BM) term. It can however be rewritten as $\ln(|\nabla\phi|/|\phi|)\phi + (\ln|\phi|)\phi$, which is a strictly homogeneous operator plus a (BM) term.

3 A Canonical Procedure for Constructing N-Particle Hierarchies

Here we develop a canonical decomposition and construction of hierarchies of N-particle time-evolution equations satisfying the separation property.

Returning to equation (6), suppose we take a particular instance of that equation, i.e., one with fixed species labels $s^{(j)}$; where the operators on the left-hand side are mixed-logarithmic homogeneous. If we have a nonzero function ϕ of the form $\phi_1\cdots\phi_r$, the ϕ_j can be extracted from ϕ only up to multiplicative constants whose product is 1. Since the left-hand side of (6) is precisely insensitive to such constants, it *can* be construed as an operator acting on ϕ. However, more is true: one can extend this operator to *arbitrary* functions, not necessarily tensor products, and thereby define F_s in such a way that *this particular instance* of (6) is true.

To see this, let G be an operator acting on functions of ℓ particle positions, producing functions of the same type. Let $m > \ell$, and let $J = (j_1, \ldots, j_\ell)$ be an ℓ-tuple of distinct elements of the set of indices $\{1, \ldots, m\}$, in increasing order. A function $\phi(x_1, \ldots, x_m)$ of m particle positions can be viewed as a parameterized family of functions $\phi_y(x_{j_1}, \ldots, x_{j_\ell})$ of ℓ particle positions, where each x_i for $i \notin \{j_1, \ldots, j_\ell\}$ is taken as a parameter y_i. Applying

G to each member of this family produces another family $G(\phi_y)(x_{j_1}, \ldots, x_{j_\ell})$, which we reinterpret back as a function $G^J(\phi)(x_1, \ldots, x_m)$. This defines a new operator G^J.

Let now J_j be the n_j-tuple of successive integers from $n_1 + n_2 + \cdots + n_{j-1} + 1$ to $n_1 + n_2 + \cdots + n_j$. Define the operator $F_s^\#$ by

$$F_s^\#(\phi) = \sum_j F_{s^{(j)}}^{J_j}(\phi) - (r-1)(p \ln |\phi| + iq \arg \phi)\phi. \tag{10}$$

We claim that $F_s^\#$ is an operator satisfying the given instance of (6). To see this, apply $F_s^\#$ to $\phi = \phi_1 \cdots \phi_r$. The term $F_{s^{(j)}}^{J_j}(\phi)$ has the operator $F_{s^{(j)}}$ acting on ϕ_j, multiplied by the product of the rest of the functions, which however depend on other variables — so these can be construed as just parameterized multipliers. Thus by the homogeneity property, letting $\hat{\phi}_j$ denote the product $\prod_{k \neq j} \phi_k$, we have

$$F_{s^{(j)}}^{J_j}(\phi) = \hat{\phi}_j F_{s^{(j)}}(\phi_j) + \hat{\phi}_j(p \ln |\hat{\phi}_j| + iq \arg \hat{\phi}_j)\phi_j.$$

Summing these to evaluate (10), and remembering that $\hat{\phi}_j \phi_j = \phi$, $\sum_j \ln \hat{\phi}_j = (r-1) \ln \phi$, and $\sum_j \arg \hat{\phi}_j = (r-1) \arg \phi$, we see that (6) is satisfied. It is easily checked that $F_s^\#$ so constructed automatically satisfies the homogeneity property.

Definition 5. We call the construction given by (10) the *canonical construction,* of $F_s^\#$ from the $F_{s^{(j)}}$, or the *canonical lifting* of the $F_{s^{(j)}}$ to $F_s^\#$.

One particular case of the canonical construction is to form $F_s^\#$ entirely from $F_{(1)}$. Regarding this, we have:

Theorem 2. *Let F be a family of mixed-logarithmic homogeneous one-particle operators with the same logarithmic indices p and q. Then the family of operators $F^\# = (F_s^\#)$ indexed by species labels, defined for $s = (s_1, \ldots, s_N)$ by*

$$F_s^\#(\phi) = \sum_j F_{(s_j)}^{(j)}(\phi) - (N-1)(p \ln |\phi| + iq \arg \phi)\phi,$$

is a tensor derivation satisfying the permutation property.

Proof: The permutation property is obvious from the construction. Consider the left-hand side of (6), and let $\phi = \phi_1 \cdots \phi_r$. We have:

$$\frac{F_{s^{(j)}}(\phi_j)}{\phi_j} = \frac{\sum_k F_{(s_k)}^{(k)}(\phi_j) - (n_j - 1)(p \ln |\phi_j| + iq \arg \phi_j)\phi_j}{\phi_j}.$$

The right-hand side can be rewritten,

$$\frac{\sum_k \hat{\phi}_j F_{(s_k)}^{(k)}(\phi_j) - (n_j - 1)(p \ln |\phi_j| + iq \arg \phi_j)\phi}{\phi}.$$

Using the homogeneity property for the operators in F, the numerator of this expression becomes

$$\sum_k F_{(s_k)}^{(k)}(\phi) - n_j(p \ln |\hat{\phi}_j| + iq \arg \hat{\phi}_j)\hat{\phi}_j \phi_j - (n_j - 1)(p \ln |\phi_j| + iq \arg \phi_j)\phi,$$

where the sum over k is still restricted to the variables of ϕ_j. Noting that $\ln|\hat{\phi}_j| + \ln|\phi_j| = \ln|\phi|$, and $\arg\hat{\phi}_j + \arg\phi_j = \arg\phi$, the sum over j of the last-displayed expression is exactly $F_s^{\#}(\phi)$, Q.E.D.

We now consider the question of uniqueness. We note that the set of tensor derivations is a linear space under component-wise operations (disregarding domain questions; for this statement to be strictly true, we must fix a domain). The logarithmic indices are linear functionals on this space. Given any derivation F, one can form the derivation $F - (F_{[1]})^{\#}$, for which all the one-particle operators are zero.

Definition 6. We call the *threshold* of a tensor derivation F the largest integer c such that for $N < c$, all the N-particle operators are zero.

Suppose that a derivation F has threshold $\ell > 1$. Since all operators in $F_{[1]}$ vanish, the logarithmic indices must be 0. By (5) all the operators in $F_{[\ell]}$ must vanish on all product functions. Starting with $F_{[\ell]}$, we can use a canonical procedure to construct higher-particle-number operators, in a manner we next describe.

Let F now be just a family, satisfying the permutation property, of strictly homogeneous ℓ-particle operators that vanish on product functions. Let $N \geq \ell$, and let s be a species index with N elements. For each ℓ-tuple $J = (j_1, \ldots, j_\ell)$ of elements of $\{1, \ldots, N\}$ in increasing order, let s_J be the ℓ-order species label $(s_{j_1}, \ldots, s_{j_\ell})$.

Theorem 3. *In the notation of the previous paragraph, let the family of operators $F^{\#} = (F_s^{\#})$, indexed by species labels, be defined for $s = (s_1, \ldots, s_N)$, $N \geq \ell$, by*

$$F_s^{\#} = \sum_J F_{s_J}^J, \tag{11}$$

and with $N < \ell$, by $F_s^{\#} = 0$. Then $F^{\#}$ is a tensor derivation satisfying the permutation property.

Proof: The permutation property is again evident from the construction. Consider once more the left-hand side of (6). As before, let J_j be the n_j-tuple of successive integers from $n_1 + n_2 + \cdots + n_{j-1} + 1$ to $n_1 + n_2 + \cdots + n_j$. We have by strict homogeneity

$$\frac{F_{s(j)}^{\#}(\phi_j)}{\phi_j} = \frac{\sum_K \hat{\phi}_j F_{s_K}^K(\phi_j)}{\phi} = \frac{\sum_K F_{s_K}^K(\phi)}{\phi},$$

where K ranges over all the ℓ-tuples of members in increasing order of J_j. Now for any ℓ-tuple J in (11) that is not comprised the members of one of the J_j, one has $F_{s_J}^J(\phi) = 0$ since the operators in F vanish on product functions. Thus the sum of these over all j is just $F_s^{\#}(\phi)/\phi$, Q.E.D.

Definition 7. We call the construction given by (11) the *canonical construction* of $F_s^{\#}$ from F, or the *canonical lifting* of F to the $F_s^{\#}$.

For a derivation F of the threshold ℓ, the derivation $F - (F_{[\ell]})^{\#}$ now has a threshold greater than ℓ. Thus we can iterate the canonical construction.

Definition 8. Let F be a tensor derivation. Define derivations $d_j F$ as follows: for $j = 1$,

$$d_1 F = (F_{[1]})^{\#};$$

having defined (up to $j = r$) $d_1F, \ldots, d_rF$, let

$$d_{r+1}F = ((F - \sum_{j=1}^{r} d_jF)_{[r+1]})^{\#}.$$

We call this the *canonical decomposition* of F, and we call the operators in $(d_jF)_{[j]}$ the *canonical generators at the threshold j* of F.

Theorem 4. *For any tensor derivation F, one has $F = \sum\limits_{j=1}^{\infty} d_jF$. The d_j are real-linear idempotents, and if d_jF is not zero, its threshold is j. Conversely, suppose for each j we are given a set of j-particle operators F_j satisfying the permutation property and the following two conditions: (a) the operators in F_1 are mixed-logarithmic homogeneous, having all the same logarithmic indices; and (b) for $j > 1$, the operators in F_j are strictly homogeneous and vanish on product functions. Then the derivation $F = \sum\limits_{j=1}^{\infty} (F_j)^{\#}$ satisfies $d_jF = (F_j)^{\#}$, and its set of canonical generators at the threshold j is precisely F_j. Furthermore, F satisfies the permutation property.*

Taking into account the construction and the earlier theorems, the proof is a series of straightforward verifications. This theorem provides an effective and easy way of constructing tensor derivations starting with the canonical generators. These are subject only to the permutation property, and conditions (a) and (b) above. The generators at the threshold ℓ represent truly new effects that come into being with ℓ particles, and are absent for any smaller number of particles.

4   Conservation of Probability

If we are to interpret the wave-function solutions as probability amplitudes, then initial data of norm 1 must continue to have norm 1 under evolution. This is true if the evolution is norm-preserving in general, and it is this property that we examine. Most examples in the literature obey this condition. If $\psi(t)$ evolves according to the equation $i\hbar\partial_t\psi = F_s(\psi)$, and if this evolution is norm preserving, then from $\partial_t(\psi(t), \psi(t)) = 0$ we deduce that $(\psi(t), F_s(\psi(t)) - (F_s(\psi(t)), \psi(t)) = 2i\,\text{Im}\,(\psi(t), F_s(\psi(t)) = 0$. Any linear hermitian operator satisfies this.

Definition 9. An operator F is *norm-hermitian*, if $\text{Im}\,(\phi, F(\phi)) = 0$ for all ϕ. Note that norm-hermiticity is a linear condition on the operator. We say that a family of operators F indexed by species labels is norm-hermitian, if each F_s is norm-hermitian.

Theorem 5. *If F is a norm-hermitian operator, then the evolution defined by $i\hbar\partial_t\psi = F(\psi)$ is norm-preserving.*

As before, the converse depends on existence and uniqueness theorems for the initial-value problem.

Now let F be a mixed-logarithmic homogeneous, norm-hermitian operator. Then we have

$$0 = \text{Im}\,(k\phi, F(k\phi)) = |k|^2\,\text{Im}\,(\phi, F(\phi)) + |k|^2||\phi||^2\,\text{Im}\,(p\ln|k| + iq\arg k)\,,$$

for all ϕ and k. But the first term after the second equality vanishes, since F is norm-hermitian. Thus we deduce that p must be real, and q pure imaginary. Moreover, if p is real and q is pure imaginary, then the operator Λ given by $\Lambda(\phi) = (p\ln\phi + iq\arg\phi)\phi$ is norm-hermitian. Suppose that G is a norm-hermitian ℓ-particle operator and let G^J be its lifting to an m-particle operator for some $m \geq \ell$ and ℓ-tuple J. Then it is easy to see that G^J is also norm-hermitian, since $\mathrm{Im}\,(\phi, G^J\phi) = \int \mathrm{Im}\,(\phi_y, G(\phi_y))\,dy$; where y stands for the variables not indexed by J, and ϕ_y is ϕ considered as a function of ℓ variables parametrized by the $m - \ell$ variables y. Taking the remarks of this paragraph into account, we see that all the canonical constructions we previously introduced preserve norm-hermiticity. Hence we have:

Theorem 6. *A tensor derivation F is norm-hermitian if and only if its canonical generators are norm-hermitian. In particular, this entails that the first logarithmic index is real and the second pure imaginary.*

This result means it is very easy to build up hierarchies of a norm-preserving evolution equation satisfying the separation property, in which truly new effects can appear as the number of particles increases. All one need do is to introduce a homogeneous norm-hermitian operator that vanishes on product functions, and to take this as a new canonical generator. This can be done even if the theory up to a given number of particles is linear; so in particular, the usual linear theories may be modified to include nonlinear effects, appearing for the first time at any particle number level, without compromising norm preservation or the separation property.

5   Examples

To get an idea of the sorts of operators that vanish identically on product functions, let us consider nonlinear differential operators of the second order not depending explicitly on the position or time coordinates, in the case $N = 2$. Such an operator has the form

$$H\Big(\phi, \frac{\partial\phi}{\partial x_i}, \frac{\partial\phi}{\partial y_j}, \frac{\partial^2\phi}{\partial x_i \partial x_j}, \frac{\partial^2\phi}{\partial x_i \partial y_j}, \frac{\partial^2\phi}{\partial y_i \partial y_j}\Big).$$

Introducing variable names for the arguments of H, we write $H(a, b_i, c_j, d_{ij}, e_{ij}, f_{ij})$. When ϕ is constrained to be a product $\phi(x, y) = \alpha(x)\beta(y)$, then the arguments of H are constrained to take on values of the form $(\alpha_0\beta_0, \alpha_i\beta_0, \alpha_0\beta_j, \alpha_{ij}\beta_0, \alpha_i\beta_j, \alpha_0\beta_{ij})$, where $\alpha_0, \beta_0, \alpha_i, \beta_j, \alpha_{ij}, \beta_{ij}$ can be given arbitrary complex values. This defines a parametrized complex algebraic variety. We must now find the generators of the ideal of complex-valued polynomials in the variables and their complex conjugates that vanish on this variety. Since the α_{ij} and β_{ij} appear each in only one component of the parameterization, generators can be chosen that do not depend on the corresponding components d_{ij} and f_{ij}. If we define $e_{00} = a, e_{0j} = c_j, e_{i0} = b_i$, and let the upper case indices I, J, K, L range over $0, 1, \ldots, d$, then the parameterization of our variety is given by $e_{IJ} = \alpha_I\beta_J$. This is equivalent to saying that e_{IJ} is a rank one or zero matrix. By standard results about determinantal ideals [15], the ideal of polynomials over the complex numbers vanishing on the variety of such matrices is generated by the order-two minors $M_{IJKL} = e_{IJ}e_{KL} - e_{IL}e_{KJ}$, which in the form of the original indices are the polynomials $ae_{ij} - b_ic_j,\ c_ie_{jk} - c_ke_{ji},\ b_ie_{jk} - b_je_{ik},$

$e_{ij}e_{kl} - e_{il}e_{kj}$. These and their complex conjugates then generate an ideal of complex-valued polynomials which vanish on the underlying real variety.

To form a complex-valued $\mathcal{C}^\infty$ function Q vanishing on this variety, one can put $Q = \sum_{IJKL} A_{IJKL} M_{IJKL} + \sum_{IJKL} B_{IJKL} \overline{M_{IJKL}}$ for some $\mathcal{C}^\infty$ coefficients A, B. If we do not admit second derivatives, then it is easy to see that for $a \neq 0$, any (a, b_i, c_j) can be written as $(\alpha_0\beta_0, \alpha_i\beta_0, \alpha_0\beta_j)$; so assuming H is continuous, there is no first-order differential operator that vanishes identically on product functions. But there are many such second-order operators. One particularly simple, rotation-invariant example corresponds to the polynomial $\mathrm{Tr}(ae_{ij} - b_ic_j)$ and is given by

$$\phi \nabla^{(1)} \cdot \nabla^{(2)} \phi - \nabla^{(1)} \phi \cdot \nabla^{(2)} \phi.$$

We should remark that in general, one would not expect any nontrivial linear operators to vanish on product functions. This is because one usually seeks operators that are continuous in some space of test functions; and in such a space, sums of product functions generally form a dense subspace. Linear operators vanishing on product functions would then vanish identically. As the above example shows, such obstructions do not pertain to nonlinear operators.

For examples of norm-hermitian generators at the threshold 2, let

$$M(\phi) = \frac{\phi \nabla^{(1)} \cdot \nabla^{(2)} \phi - \nabla^{(1)} \phi \cdot \nabla^{(2)} \phi}{\phi^2}, \tag{12}$$

and take any operator of the form $(k_1 \,\mathrm{Re}\, M(\phi) + k_2 \,\mathrm{Im}\, M(\phi))\phi$ for real k_1, k_2. These operators are similar in appearance to terms in one-particle operators introduced by Doebner and Goldin [3] for Galileian-invariant theories.

The equations of Doebner and Goldin have the form

$$i\hbar\partial_t\psi = -(\hbar^2/2m)\nabla^2\psi + iD\hbar \left[\frac{1}{2}\frac{\nabla^2(\bar{\psi}\psi)}{\bar{\psi}\psi}\right]\psi + R(\psi)\psi,$$

where D is a real physical constant, and

$$R(\psi) = \hbar D \left\{ \lambda_1 \,\mathrm{Re}\, \frac{\nabla^2\psi}{\psi} + \lambda_2 \,\mathrm{Im}\, \frac{\nabla^2\psi}{\psi} + \lambda_3 \,\mathrm{Re}\, \frac{(\nabla\psi)^2}{\psi^2} + \lambda_4 \,\mathrm{Im}\, \frac{(\nabla\psi)^2}{\psi^2} + \lambda_5 \,\mathrm{Im}\, \frac{|\nabla\psi|^2}{|\psi|^2} \right\}.$$

The corresponding equations are Galileian invariant if $\lambda_2 + \lambda_4 = 0$ and $\lambda_1 + \lambda_3 = \lambda_5$. By this is meant that if $\psi(t, x)$ is a solution, then so is $\hat{\psi}(t, x) = e^{-i\theta(t,x)}\psi(t, x + vt)$ where $\theta(t, x) = \frac{1}{\hbar}\left(\frac{1}{2}mv^2t + mvx\right)$. It is easily checked that the corresponding canonically-constructed N-particle equations continue to be Galileian invariant, using the transformation $\psi(t, x_1, \ldots, x_N) \mapsto e^{-i(\theta_1(t,x_1) + \cdots + \theta_N(t,x_N))}\psi(t, x_1 + vt, \ldots, x_N + vt)$.

Furthermore addition of two-particle canonical generators that take the form

$$(k_1 \,\mathrm{Re}\, M(\psi) + k_2 \,\mathrm{Im}\, M(\psi))\psi,$$

where M is given by (12), does not spoil the invariance of the N-particle equations under the same transformation. Thus we can maintain in a complete hierarchy of multi-particle equations the Galileian invariance, the separation property, and norm-preservation, while introducing truly new physical effects at the two-particle threshold.

The question of Galileian invariance for nonlinear one-particle Schrödinger equations was investigated by Fushchych and Cherniha [16–18], which found a class of equations and some of their exact solutions. The general problem of symmetries of evolutions obeying the separation property will be taken up in a forthcoming paper by one of the authors (Svetlichny).

Acknowledgments

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