On the Spectral Theory of Operator Pencils in a Hilbert Space

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Abstract

Consider the operator pencil \( L_\lambda = A - \lambda B - \lambda^2 C \), where \( A \), \( B \), and \( C \) are linear, in general unbounded and nonsymmetric, operators densely defined in a Hilbert space \( H \). Sufficient conditions for the existence of the eigenvalues of \( L_\lambda \) are investigated in the case when \( A \), \( B \) and \( C \) are \( K \)-positive and \( K \)-symmetric operators in \( H \), and a method to bracket the eigenvalues of \( L_\lambda \) is developed by using a variational characterization of the problem (i) \( L_\lambda u = 0 \). The method generates a sequence of lower and upper bounds converging to the eigenvalues of \( L_\lambda \) and can be considered an extension of the Temple-Lehman method to quadratic eigenvalue problems (i).

1 Introduction

Let \( H \) be a separable complex Hilbert space with the inner product and norm

\[
(x, y), \|x\| = (x, x)^{1/2}, \quad (x, y \in H)
\]

and consider in \( H \) the nonlinear eigenvalue problem

\[
Ax - \lambda Bx - \lambda^2 Cx = 0
\]

where \( A \) and \( C \) are \( K \)-p.d. operators with \( D_C \supseteq D_A \), \( D_A \) is dense in \( H \), and \( B \) is an operator with \( D_B \supseteq D_C \). Recall [1–3] that by the definition of \( A \) and \( C \) there exists a closable operator \( K \) with \( D_K \supseteq D_C \) mapping \( D_A \) onto a dense subset \( KD_A \) of \( H \) and positive constants \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) such that

\[
(Ax, Kx) \geq \alpha_1 \|x\|^2, \quad (x \in D_A)
\]

\[
\|Kx\|^2 \leq \alpha_2 (Ax, Kx), \quad (x \in D_A)
\]

\[
(Cx, Kx) \geq \beta_1 \|x\|^2, \quad (x \in D_C)
\]

\[
\|Kx\|^2 \leq \beta_2 (Cx, Kx), \quad (x \in D_C)
\]

The class of \( K \)-p.d. operators \( \{P\} \) contains, among others, the following families of mappings:

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(a) Positive definite operators; in this case $K$ is the identity map or, if $P$ is also self-adjoint, $K$ can be any root of $P$.

(b) Closeable and densely invertible operators; in this case we let $K = P$.

(c) The operator $P$ of the form $P = -S^{2j+1}$ or $P = S^{2j+2}$ where for some $i$, $0 \leq i < j$, the operator $S^{2(j+i+1)}$ is positive definite; in this case we let $K = S^{2i+1}$ or $K = S^{2i+2}$, provided that $K$ so defined is closable and $KD_P$ is dense in $H$. To this class belong, in particular, ordinary differential operators of odd and even order and weakly elliptic partial differential operators of odd and even order which in general are not self-adjoint [2].

(d) A subclass of bounded and unbounded symmetrizable operators investigated by a number of authors [2, 4].

Let $D[A]$ be the set $D_A$ endowed with the new metric

$$(x, y)_A = (Ax, Ky), \quad ||x||_A^2 = (x, x)_A, \quad (x, y \in D_A)$$

and denote by $H_A$ the completion of $D[A]$ in the metric (7). Similarly, let $D[C]$ be the set $D_C$ with the metric

$$(x, y)_C = (Cx, Ky), \quad ||x||_C^2 = (x, x)_C, \quad (x, y \in D_C)$$

and define $H_C$ to be the completion of $D[C]$ in the metric (8). One can show that the space $H_A$ is contained in $H$ in the sense of uniquely identifying the elements of $H_A$ with certain elements in $H$ and clearly, since $C$ is $K$-p.d., the above assertion is valid also for the space $H_C$, i.e., $H_C \subseteq H$. Let $H_1 = H \times H_C$ be the Cartesian product space, with the norm and inner product defined by

$$(u, v)_1 = (x, p) + (y, q)_C$$

and

$$||u||_1 = (u, u)_1^{1/2} = \left(||x||^2 + ||y||_C^2\right)^{1/2}.$$  

Clearly, $H_1$ is a Hilbert space and, since $H_C$ is a subset of $H$, it follows that $H_1 \subseteq H \times H$ in the sense mentioned above. Now, let $T : D_A \times D[C] \subseteq H_1 \rightarrow H_1$ be the operator matrix

$$T = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ y \end{pmatrix}, \quad (u = \begin{pmatrix} x \\ y \end{pmatrix} \in D_A \times D[C]).$$

Similarly, let us define in $H_1$ the operators

$$S = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix}, \quad S : D_B \times D[C] \subseteq H_1 \rightarrow H_1,$$

$$\hat{K} = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}, \quad \hat{K} : D_K \times D[C] \subseteq H_1 \rightarrow H_1.$$  

\footnote{An operator $P$ will be called invertible if it has a bounded inverse, densely invertible if it is invertible and its range $R_P$ is dense in $H$, and continuously invertible if it is densely invertible and $R_P = H$.}
Observe that the quadratic eigenvalue problem (2) is equivalent to the system

\[ Ax - \lambda B x - \lambda C y = 0 \]
\[ y - \lambda x = 0 \]  \hspace{1cm} (14)

which, in view of (11) and (12), is equivalent to the linear equation

\[ Tu - \lambda Su = 0 \]  \hspace{1cm} (15)

in the sense that if \( x_i \) is a solution of (2) corresponding to \( \lambda = \lambda_i \), then \( u_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \) with \( y_i = \lambda_i x_i \) is a solution of (15) and, conversely, if \( u_i \) is a solution of (15) corresponding to \( \lambda = \lambda_i \), then \( y_i = \lambda_i x_i \) and \( x_i \) is a solution of (2).

**Proposition 1.** The operator \( T \) defined by (11) is \( K \)-p.d. in the space \( H_1 = H \times H_C \); i.e., \( T \) satisfies the following conditions:

(a) \( D_T \) is dense in \( H_1 \).

(b) \( D_K \supseteq D_T \) and \( KD_T \) is dense in \( H_1 \).

(c) \( K \) is closable in \( H_1 \).

(d) There exist positive constants \( \gamma_1, \gamma_2 \) such that

\[ (Tu, \hat{K}u)_1 \geq \gamma_1 \|u\|_1^2, \quad (u \in D_T) \]  \hspace{1cm} (16)

\[ \|\hat{K}u\|_1^2 \leq \gamma_2 (Tu, \hat{K}u)_1, \quad (u \in D_T). \]  \hspace{1cm} (17)

**Proof.** (a) Let \( u = \begin{pmatrix} x \\ y \end{pmatrix} \) be an arbitrary element in \( H_1 = H \times H_C \). Since \( D_A \) is dense in \( H \), there exists a sequence \( \{x_n\} \subset D_A \) which converges to \( x \) in the \( H \)-metric. Similarly, since \( D_C \) is dense in \( H_C \), there exists a sequence \( \{y_n\} \subset D_C \) which converges to \( y \) in the \( H_C \)-metric. Hence, if we define a sequence in \( D_T = D_A \times D[C] \) by \( u_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \), then

\[ \lim_{n \to \infty} \|u_n - u\|_1^2 = \lim_{n \to \infty} (\|x_n - x\|^2 + \|y_n - y\|^2_2) = 0. \]  \hspace{1cm} (18)

(b) By definition, \( KD_T = KD_A \times D[C] \) where \( KD_A \) is dense in \( H \). Hence, using a similar argument as in part (a), one can show that \( KD_T \) is dense in \( H_1 \). Moreover, since \( D_K \supseteq D_A \), it follows that

\[ D_K = D_K \times D[C] \supseteq D_A \times D[C] = D_T. \]  \hspace{1cm} (19)

(c) Let \( u_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \) be a sequence in \( D_K \), and \( f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \) an element in \( H_1 \) such that the following conditions hold:

\[ \lim_{n \to \infty} \|u_n\|_1 = 0, \]  \hspace{1cm} (19)

\[ \lim_{n \to \infty} \|\hat{K}u_n - f\|_1 = 0. \]  \hspace{1cm} (20)

From (20) we obtain

\[ \lim_{n \to \infty} \sqrt{\|Kx_n - f_1\|^2 + \|y_n - f_2\|^2_2} = 0 \]  \hspace{1cm} which implies that

\[ Kx_n \to f_1 \in H. \]  \hspace{1cm} (21)
$y_n \to f_2 \in H_C,$ \hfill (22)

On the other hand, from (19) we deduce that

$x_n \to 0$ in the $H$-norm, and

$y_n \to 0$ in the $H_C$-norm. \hfill (23)

In view of (21) and (23) it follows that $f_1 = 0$, since $K$ is closable in $H$. Moreover, from (22) and (24) it follows that $f_2 = 0$. Hence, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$, and $\hat{K}$ is closable in $H_1$.

(d) If $u = \begin{pmatrix} x \\ y \end{pmatrix} \in D_T$, then $(Tu, \hat{K}u) = (Ax, Kx) + (y, y)_C$ and, in view of (3), we obtain the inequality

$(Tu, \hat{K}u)_1 \geq \alpha_1 \|x\|^2 + \|y\|^2_C$. \hfill (25)

Let $\gamma_1 = \min\{\alpha_1, 1\}$; then from (25) and (10) it follows that

$(Tu, \hat{K}u)_1 \geq \gamma_1 \|u\|^2_1, \quad (u \in D_T)$. \hfill (26)

Since $\|\hat{K}u\|^2_1 = (Kx, Kx) + (y, y)_C$, it follows from (4) and (11) that

$\|\hat{K}u\|^2_1 \leq \alpha_2 (Ax, Kx) + (y, y)_C \leq \gamma_2 (Tu, \hat{K}u)_1, \quad (u \in D_T)$, \hfill (27)

where $\gamma_2 = \max\{\alpha_2, 1\}$. \hfill \blacksquare

Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$ and $v = \begin{pmatrix} p \\ q \end{pmatrix}$ be elements of the space $D_T = D_A \times D_C \subseteq H \times H_C = H_1$ and let us introduce in $D_T$ a new norm and inner product

$$(u, v)_2 = (Tu, \hat{K}v)_1 = (x, p)_A + (y, q)_C \hfill (28)$$

$$\|u\|_2 = \sqrt{\|x\|^2_A + \|y\|^2_C}. \hfill (29)$$

Define by $D[T]$ the linear set $D_T$ endowed with the metric $\|\cdot\|^2_2 = (\cdot, \cdot)_2$, and observe that

$$D[T] = D[A] \times D[C]. \hfill (30)$$

In view of (16), (17) and the fact that $T$ is $K$-p.d. in $H_1$, we have the inequalities

$$\|u\|_2 \geq \sqrt{\gamma_1} \|u\|_1, \quad (u \in D_T) \hfill (31)$$

$$\|\hat{K}u\|_1 \leq \sqrt{\gamma_2} \|u\|_2, \quad (u \in D_T) \hfill (32)$$

Clearly, $D_T$ satisfies all the properties of a Hilbert space, with the possible exception of completeness. Let us denote by $H_2$ the completion of $D[T]$ in the metric (29).

**Proposition 2.**

(a) $H_2 = H_A \times H_C$.

(b) $H_2$ is contained in $H_1$ in the sense of identifying uniquely the elements from $H_2$ with certain elements in $H_1$. 

(c) $\hat{K}$ can be extended to a bounded operator $\hat{K}_0$ mapping all of $H_2$ to $H_1$ such that $\hat{K} \subset \hat{K}_0 \subset \overline{\hat{K}}$, where $\overline{\hat{K}}$ denotes the closure of $\hat{K}$ in $H_1$.

(d) $T$ has a unique closed $\hat{K}_0$-p.d. extension $T_0$ such that $T_0 \supseteq T$, $T_0$ has a bounded inverse $T_0^{-1}$ defined on all of $H_1 = R_{T_0}$, and the inequalities (31) and (32) remain valid in $H_2$ in the form

\begin{align*}
||u||_2 &\geq \sqrt{\gamma_1}||u||_1, \quad (u \in H_2) \\
||\hat{K}_0u||_1 &\leq \sqrt{\gamma_2}||u||_2, \quad (u \in H_2).
\end{align*}

**Proof.** The proof of part (a) follows from (30) and the fact that $H_A$ and $H_C$ are the completions of the spaces $D[A]$ and $D[C]$ in the norms $|| \cdot ||_A$ and $|| \cdot ||_C$, respectively. By Proposition 1, the operator $T$ is $\hat{K}$-p.d. in $H_1$. Hence, the proof of parts (b), (c), and (d) can be derived from Lemma 1.2 of Petryshyn [3], provided the spaces $H_2, H_1$ and the operator $\hat{K}$ in Proposition 2 are identified with $H_0, H$, and $K$, in Lemma 1.2, respectively.

In the sequel we shall assume, when necessary, that the operators $\hat{K}$ and $T$ have already been extended and the notation $T_0$ and $\hat{K}_0$ will not be used. Note that in applications it is often not necessary to extend the operators $T$ and $\hat{K}$.

## 2 The equivalent linear problem

$Tu - \lambda Su = 0$

**Definition 1.** The quadratic eigenvalue problem

\[ Ax - \lambda Bx - \lambda^2 Cx = 0, \]

where $A$ and $C$ are $K$-p.d. with $D[A] \subseteq D[C] \subseteq D_B$ and $B$ is $K$-symmetric on $D_C$, i.e.,

\[ (Bx, Ky) = (Kx, By), \quad (x, y \in D_C) \]

will be called $K$-real.

**Proposition 3.** If the quadratic eigenvalue problem (35) is $K$-real in $H$, then the equivalent linear problem

\[ Tu - \lambda Su = 0 \]

(37)

defined by (11)-(13) is $\hat{K}$-real in $H_1 = H \times H_C$, i.e. $T$ is $\hat{K}$-p.d. and $S$ is $\hat{K}$-symmetric on $D_T$.

**Proof.** In view of Proposition 1, only the $\hat{K}$-symmetry of $S$ needs to be verified. To this end let $u = \left( \begin{array}{c} x \\ y \end{array} \right)$ and $v = \left( \begin{array}{c} p \\ q \end{array} \right)$ be elements in $D_T \subseteq H_1$ and note that

\[ (Su, \hat{K}v)_1 = (Bx + Cy, Kp) + (x, q)_C = (Bx, Kp) + (Cy, Kp) + (Cx, Kq) \]

(38)

Since by definition the operators $B$ and $C$ are $K$-symmetric on $D_A \subseteq H$, the above equation yields the identity

\[ (Su, \hat{K}v)_1 = (Kx, Bp + Cq) + (Cy, Kp) = (\hat{K}u, Sv)_1, \quad (u, v \in D_T) \]

(39)

which proves the $\hat{K}$-symmetry of $S$ on $D_T$. \( \blacksquare \)
Let us assume that the eigenvalue problem (35) is $K$-real, which implies that problem (37) is $\hat{K}$-real. A value of the complex parameter $\lambda$ for which (37) has a nontrivial solution $u \in D_T$ will be called an eigenvalue of (37), and $u$ its corresponding eigenfunction. The set of all eigenvalues of (37) will be denoted by $\rho \sigma(37)$ and called the point spectrum of (37). By the multiplicity of $\lambda$ we shall mean the number of linearly independent eigenfunctions which correspond to $\lambda$. Since $T, S$ are $\hat{K}$-symmetric, it follows [5] that the eigenvalues of (37) are real, and the eigenfunctions $u_1, u_2$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2$ are orthogonal in the sense that $(Tu_1, \hat{K} u_2)_1 = 0$. Since the space $H_1$ is separable, it follows that the point spectrum of (37) is countable.

Suppose the operators $K$ and $L_K \equiv A - \lambda B - \lambda^2 C$ are closed with $D_K = D_C$, and that $L_\lambda : D_K \subseteq H \to H$ is a bijection for all $\lambda$, except possibly for a discrete set of eigenvalues of the problem $(A - \lambda B - \lambda^2 C)x = 0$. Under the above assumptions, it is not difficult to show that the equivalent linear problem $Tu - \lambda Su = 0$ in $H_1 = H \times H_C$ satisfies the following conditions:

$(\alpha)$: The operator $G_\lambda = T - \lambda S : D_T \subseteq H_1 \to H_1$ is continuously invertible for all $\lambda \notin \rho \sigma(37)$.

$(\beta)$: The spectrum $\sigma(N)$ of the operator $N = T^{-1}S : D[T] \subseteq H_2 \to D[T]$, contains only eigenvalues of finite multiplicity with zero as its sole possible limit point.

Let $\rho \sigma(L_\lambda) = \{\lambda_i : i = 1, 2, \ldots\}$ denote the point spectrum of the operator $L_\lambda$, with the eigenvalues ordered according to increasing magnitude and repeated as many times as their multiplicity indicates. Let $\{x_i : i = 1, 2, \ldots\}$ be the set of corresponding eigenfunctions, normalized in the sense that $||x_i||_A^2 + \lambda_i^2 ||x_i||_C^2 = 1$. Then, using certain results from the theory of linear $K$-real eigenproblems $Tu - \lambda Su = 0$, we may derive the following theorems, which extend the corresponding results [6–8] obtained for the case when $C$ is the identity operator and $A, B$ are self-adjoint, positive definite, or compact operators. (Related results, under different assumptions on the operators $A, B, C$, have been obtained by other authors) [9–15].

**Theorem 1.** Assume that the eigenproblem (35) is $K$-real, that $L_\lambda : D_A \subseteq H$ is a bijection for all $\lambda \notin \rho \sigma(L_\lambda)$ and that the operators $L_\lambda$ and $K$ are closed with $D_K = D_C$. Then the eigenvalues and eigenfunctions of problem (35) have the variational characterization

$$\frac{1}{|\lambda_n|} = \sup_{(x,y) \in D_A \times D_C} \{|E(x,y) : (x, x_i)_A + \lambda_i(y, x_i)_C = 0, \ 1 \leq i \leq n - 1\} = E(x_n, \lambda_n x_n),$$

where $E(x,y) = \frac{(Bx, Kx) + 2Re(Cx, Ky)}{(Ax, Kx) + (Cy, Ky)}$.

Moreover, the eigenvalues found by this variational process exhaust entirely the set $\rho \sigma(L_\lambda)$.

**Proof.** By hypothesis the linearized eigenproblem (37) is $\hat{K}$-real and satisfies conditions $(\alpha)$ and $(\beta)$. It follows from the theory of linear $K$-real eigenproblems [3, 5] that the eigenpairs $(\lambda_i, u_i)$ of problem (37), normalized in the sense $||u_i||_2 = 1$, satisfy the variational principle

$$\frac{1}{|\lambda_n|} = \sup_{u \in D_T} \left\{ \frac{|(Su, \hat{K} u)_1|}{(Tu, \hat{K} u)_1} : (Tu, \hat{K} u)_1 = 0, \ 1 \leq i \leq n - 1 \right\} =$$
and the eigenvalues determined by (40a) exhaust entirely the set $p\sigma(37)$. Thus, the validity of the last assertion of Theorem 1 follows from the fact that $p\sigma(37) = p\sigma(L\lambda)$. If we let $u = (x, y)^T$, $u \in D(T) = D_A \times D_C$, then expanding the inner products in (40a) and using the $K$-symmetry property of the operator $C$, we obtain the expressions

$$(Su, \hat{K}u)_1 = (Bx, Kx) + (Cy, Ky) = (Bx, Kx) + 2\Re(Cx, Ky)$$

$$(Tu, \hat{K}u)_1 = (Ax, Kx) + (Cy, Ky)$$

$$||u||_2^2 = ||x_i||_A^2 + \lambda_i^2 ||x_i||_C^2$$

$$||u||_2^2 = ||x_i||_A^2 + \lambda_i^2 ||x_i||_C^2$$

Substituting the above into (40a) yields the variational formula (40).

**Lemma 1.** Assume the hypothesis of Theorem 1.

(a) Suppose $S$ and $S^+$ are $\hat{K}$-symmetric operators, $T$ is $\hat{K}$-p.d., and $|(S^+u, \hat{K}u)| \geq |(Su, \hat{K}u)|$ for $u \in D_T$. Then the eigenvalues $\lambda_+^i$ and $\lambda_i$ of the corresponding eigenproblems $Tu - \lambda^+ S^+ u = 0$ and $Tu - \lambda S u = 0$ satisfy the inequality $|\lambda_+^i| \leq |\lambda_i|$, $i=1,2,...$

(b) Suppose that $T$ and $T^*$ are $K$-p.d. operators with $D_T = D_T^*$, $S$ is $K$-symmetric on $D_T$, and $$(T^*u, \hat{K}u) \geq (Tu, \hat{K}u)$$

for $u \in D_T$. Then the eigenvalues $\lambda_*^i$ and $\lambda_i$ of the corresponding eigenproblems $Tu - \lambda S u = 0$ and $T^* u - \lambda^* S u = 0$ satisfy the inequality $|\lambda_*^i| \geq |\lambda_i|$, $i=1,2,...$

**Proof.** The proof of parts (a) and (b) is a direct consequence of the variational principle (40) in Theorem 1.

**Theorem 2.** Assume the hypothesis of Theorem 1 and let $\{u_i : 1, 2, \ldots\}$ be the set of eigenfunctions, orthonormal in $H_2$, of the $\hat{K}$-real eigenproblem (37). If $u \in D_T$, then $T^{-1}Su$ has the expansion

$$T^{-1}Su = \sum_{i=1}^{\infty} (Su, \hat{K}u_i)_1 u_i$$

which converges in the $H_1$ and $H_2$-norm.

**Proof.** The result follows directly from the corresponding eigenfunction expansion theorem [3, 5] for linear $K$-real eigenvalue problems.

### 3 Iterative method

Let $f_0 = \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right)$ be an element in $D_T$ such that $f_0 \notin N(S)$ (the null space of $S$), and denote by $f_k = \left( \begin{array}{c} x_k \\ y_k \end{array} \right)$ the iterant at the $k$-th step of our process; then the succeeding iterant $f_{k+1}$ is obtained by solving the equation $Tf_{k+1} = Sf_k$, i.e.,

$$\left( \begin{array}{cc} A & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{c} x_{k+1} \\ y_{k+1} \end{array} \right) = \left( \begin{array}{cc} B & C \\ I & 0 \end{array} \right) \left( \begin{array}{c} x_k \\ y_k \end{array} \right), \quad (k \geq 0).$$
Now, let us determine the constants
\[ a_k = (S_f k, \hat{K} k) = (B x_k, K x_k) + (C y_k, K x_k) + (C x_k, \hat{K} y_k) \]
(0 \leq i \leq k, k = 1, 2, \ldots). \quad (43)

Note that the values of \((S_f k, \hat{K} k)\) depend on \(k\) but not on \(i\), since from the \(\hat{K}\)-symmetry of \(S\) and \(T\) it follows that \((S_f k, \hat{K} f_0) = (S_f k, \hat{K} f_1) = \ldots = (S_f k, \hat{K} f_1)\). Also, note that the elements of the sequence \(\{f_k\}\) cannot vanish, since \(f_0 \notin N(S)\) implies that \(f_n \notin N(S)\) for \(n \geq 0\). Indeed, if \(S f_n \neq 0\) for \(n < k\), then \(f_k = T^{-1} S f_{k-1} \neq 0\), and from the identity \((S_f k, \hat{K} f_{k-1}) = (\hat{K} f_k, S f_{k-1}) = (T f_k, \hat{K} f_1) > 0\) it follows that \(S f_k \neq 0\). Thus, by induction, it follows that \(f_n \notin N(S)\) for all \(n \geq 0\).

Let \(H^2_2\) be the space spanned by the eigenfunction \(u_i\) and denote by \((H^2_2)^{\perp}\) the orthogonal complement of \(H^2_2\) in \(H_2\).

**Proposition 4.** Let \(c_i = (f_0, u_i), i = 1, 2, \ldots\) be the Fourier coefficients of \(f_0\) with respect to the orthonormal set of eigenfunctions \(\{u_i\}\) in \(H_2\). Then,

(a) \(f_k\) may be represented by the following series, converging in the \(H_1\) and \(H_2\) metrics:

\[ f_k = \sum_{i=1}^{\infty} c_i \lambda_i^{-k} u_i, \quad (k = 0, 1, \ldots) \quad (44) \]

(b) the constants \(a_k\), determined by (43), are of the form

\[ a_k = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-(k+1)}, \quad (k = 0, 1, \ldots) \]

**Proof.** (a) Applying Theorem 2 we may express \(f_k\) in the form

\[ f_k = T^{-1} S f_{k-1} = \sum_{i=1}^{\infty} (S f_k k, \hat{K} u_i) u_i, \quad (k = 1, 2, \ldots) \]

where the series converges in the \(H_1\) and \(H_2\) metrics. Now, let us show that the following identity is valid

\[ (S f_k k, \hat{K} u_i) = c_i \lambda_i^{-k}, \quad (k = 1, 2, \ldots) \]

(47)

For \(k = 1\) using the \(\hat{K}\)-symmetry of \(S\) and \(T\), we obtain

\[ (S f_0, \hat{K} u_i) = (\hat{K} f_0, S u_i) = \lambda_i^{-1} (\hat{K} f_0, T u_i) = \lambda_i^{-1} (T f_0, \hat{K} u_i) = c_i \lambda_i^{-1}. \]

Suppose (47) is valid for \(n < k\), then

\[ (S f_k, \hat{K} u_i) = \lambda_i^{-1} (\hat{K} f_k, S u_i) = \lambda_i^{-1} (S f_{k-1}, \hat{K} u_i) = c_i \lambda_i^{-(k+1)}. \]

Hence, identity (47) is valid by induction and substituting it into (46) completes the proof of part (a).

(b) Recall that the operator \(\hat{K}\), understood in the extended sense, is a continuous mapping from \(H_2\) into \(H_1\) and that the series (44) is convergent in the \(H_1\) and \(H_2\) metrics. Thus, applying the expansion (44) to the last term in the identity

\[ a_k = (S f_k, \hat{K} f_0) = (T f_{k+1}, \hat{K} f_0) = (\hat{K} f_{k+1}, T f_0), \]
we obtain
\[
a_k = \sum_{i=1}^{\infty} c_i \lambda_i^{-(k+1)} (K u_i, T f_0)_1 = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-(k+1)}, \quad (k = 0, 1, \ldots). \tag{48}
\]

Let \( w_k = a_{2k-1}/a_{2k+1} \) and note that by applying (45) we may express \( w_k \) in the form
\[
w_k = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2k} / \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2(k+1)}, \quad (k = 1, 2, \ldots). \tag{49}
\]

**Theorem 3.** Assume the hypothesis of Theorem 1 and suppose that \( |\lambda_r| < |\lambda_{r+1}| \) for some positive integer \( r \). If \( f_0 \) is chosen from the space
\[
f_0 \in D[T] \cap [\cap_{i=1}^{r-1} (H_2^f)^{-1}], \quad f_0 \notin (H_2^s)^{-1}, \quad r \geq 1,
\]
then the following statements are true:
(a) the sequence \( \{\sqrt{w_k}\} \) converges monotonically from above to \( |\lambda_r| \),
(b) \( s_k = \lambda_r^{2k} f_{2k}, k = 1, 2, \ldots \) converges in the \( H_2 \)-metric to an eigenfunction \( c_r u_r \in H_2^s \).

**Proof.** (a) To show monotonicity of the sequence \( \{w_k\} \), let \( z_k \in D_T \) be defined by
\[
z_k = a_{2k+3} f_k - a_{2k+1} f_{k+2}, \quad (k = 1, 2, \ldots)
\]

Then, \( 0 \leq (T z_k, \hat{K} z_k)_1 = a_{2k+3} (a_{2k+3} a_{2k-1} - a_{2k+1}^2) \)
which yields
\[
0 \leq (a_{2k-1}/a_{2k+1}) - (a_{2k+1}/a_{2k+3}) \equiv w_k - w_{k+1}, \quad (k = 1, 2, \ldots)
\]

To prove convergence, we may use (48) to express \( w_k = a_{2k+1}/a_{2k-1} \) in the form
\[
w_k = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2k} / \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2(k+1)}, \quad (k = 1, 2, \ldots). \tag{51}
\]

Using the simplified notation, \( \Lambda_i = \lambda_i^2 \), and the fact that by hypothesis \( c_1 = c_2 = \ldots = c_{r-1} = 0 \), we deduce from (51) the expression
\[
w_k = \Lambda_r P(k) / Q(k), \quad (k = 1, 2, \ldots), \tag{52}
\]
where \( P(k) \) and \( Q(k) \) are the series
\[
P(k) = \sum_{i=r}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^k, \quad Q(k) = \sum_{i=r}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^{k+1}
\]

From Bessel’s inequality \( \sum_{i=r}^{\infty} |c_i|^2 \leq ||f_0||^2_2 \) and the fact that \( (\Lambda_r/\Lambda_i) < 1 \) for \( i > r \), it follows that the series \( P(k) \) and \( Q(k) \) are uniformly convergent with respect to the parameter \( k \), and their difference may be expressed in the form
\[
P(k) - Q(k) = \sum_{i=r}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^k - (\Lambda_r/\Lambda_i)^{k+1} \leq \sum_{i=r+1}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^k
\]
\[ \leq (\Lambda_r/\Lambda_{r+1})^k ||f_0||_2^2, \quad (k \geq 1). \]  

Since \( P(k) \geq Q(k) \geq |c_r|^2 > 0 \), it follows from (53) that \( |P(k) - Q(k)| \to 0 \) and \( P(k)/Q(k) \to 1 \) as \( k \to \infty \). Therefore, from (52) it follows that \( w_k \) converges to \( \Lambda_r = \lambda_r^2 \).

(b) By Proposition 4, Eq.(44), the elements of the sequence \( s_k = \lambda_r^{2k} f_{2k}, k = 1, 2, \ldots \) may be represented by the series

\[
s_k = \sum_{i=r}^{\infty} c_i (\Lambda_r/\Lambda_i)^k u_i, \quad (k = 0, 1, \ldots)
\]

convergent in \( H_1 \) and \( H_2 \)-metrics. Hence, due to the orthonormality of the eigenvectors \( u_i \) in \( H_2 \), \( i = 1, 2, \ldots \), it follows that

\[
||s_k - c_r u_r||_2^2 = \sum_{i=r+1}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_{i+1})^{2k}.
\]

Applying Bessel’s inequality and the fact that by hypothesis \( \Lambda_r < \Lambda_{r+1} \leq \Lambda_{r+2} \leq \ldots \), we obtain the error estimate

\[
||s_k - c_r u_r||_2^2 \leq ||f_0||_2^2 (\Lambda_r/\Lambda_{r+1})^{2k}, \quad (k = 1, 2, \ldots).
\]

Thus, it follows that the sequence \( \{s_k\} \) converges in the \( H_2 \)-metric to an eigenfunction \( c_r u_r \in H^2_r \), with the error estimate given above.

Now, let us assume that a lower bound \( l_{r+1} \) for the eigenvalue \( |\lambda_{r+1}| \) can be determined by some method such as, for example, suggested by Lemma 1. Then, using the iterative process (43), we can derive a sequence of lower bounds that converges to \( |\lambda_r| \).

**Theorem 4.** Assume the hypothesis of Theorem 1. If \( l_{r+1} \) is a lower bound for \( |\lambda_{r+1}| \) such that for some positive integer \( N \) we have \( \sqrt{w_N} \leq l_{r+1} \leq |\lambda_{r+1}| \), then

\[ \Lambda_r \geq (l_{r+1}^2 - w_k)w_{k+1}/(l_{r+1}^2 - w_{k+1}) \]

for \( k \geq N \) and the sequence of lower bounds converges to \( \Lambda_r \) as \( k \to \infty \).

**Proof.** The proof of the above theorem is based on the corresponding results for linear K-real eigenvalue problems (see [5], p.207).

Theorems 3-4 allow us to bracket the eigenvalues of a quadratic eigenvalue problem (35) \( L_\lambda x = 0 \) by a procedure which is similar to the Temple-Lehman method for linear eigenvalue problems \( Mu - \Lambda Nu = 0 \). In that sense the above results may be considered an extension of the Temple-Lehman method to nonlinear (quadratic) eigenvalue problems (35), where \( A,B,C \) are symmetrizable operators in \( H \). Important extensions and applications of the Temple-Lehman method to linear problems \( Mu - \Lambda Nu = 0 \), where \( M \) and \( N \) are partial differential operators, may be found in the work of F. Goerisch and H. Haunhorst [16].
References


