Nonlinear Mathematical Physics 1996, V.3, N 1–2, 63–67.

Conditional and Nonlocal Symmetry of Nonlinear Heat Equation

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1 Conditional symmetry

We investigate conditional symmetry in three directions. The first direction is a research of the Q-conditional symmetry. The second direction is studying conditional symmetry when an algebra of invariance is known and an additional condition is unknown. The third direction is the investigation of the conditional symmetry in the case where a known additional condition differs from $Qu = 0$.

We describe these directions by example of the nonlinear heat equation

$$u_0 + \vec{\nabla}\left(f(u)\vec{\nabla}u\right) = h(u).$$

For convenience, we consider the equivalent equation

$$H(u)u_0 + \triangle u = F(u), \tag{1}$$

where $u = u(x), x = (x_0, \vec{x}) \in R_{1+n}, u_0 = \dfrac{\partial u}{\partial x_0}$.

Theorem 1. *When $n = 1$, the equation (1) is Q-conditional invariant with respect to an operator*

$$Q = A\,(x,u)\,\partial_0 + B\,(x,u)\,\partial_1 + C\,(x,u)\,\partial_u, \tag{2}$$

if the functions A, B, C are solutions of the following system of differential equations.

Case 1.

$$\begin{aligned}
&A = 1, \qquad B_{uu} = 0, \qquad C_{uu} = 2\left(B_{1u} + HBB_u\right),\\
&3B_uF = 2\left(C_{1u} + HB_uC\right) - \left(HB_0 + B_{11} + 2HBB_1 + \dot{H}BC\right),\\
&C\dot{F} - \left(C_u - 2B_1\right)F = HC_0 + C_{11} + 2HCB_1 + \dot{H}C^2;
\end{aligned} \tag{3}$$

Case 2.

$$\begin{aligned}
&A = 0, \qquad B = 1,\\
&C\dot{F} - \left(C_u + \tfrac{\dot{H}}{H}C\right)F = HC_0 + C_{11} + 2CC_{1u} + C_{1u}^2 - C\tfrac{\dot{H}}{H}\left(CC_u + C1\right).
\end{aligned} \tag{4}$$

We failed to find the general solution of the system (3), (4). But even with the help of a particular solution of these systems, we can find a whole variety of operators which cannot be found by the Lie method. Some of these operators are given below.

I. At $F(u)=0, \quad H(u)=\frac{1}{u}$, the equation

$$u_0+uu_{11}=0 \tag{5}$$

is known as the Heisenberg ferromagnetic equation. The following operators were obtained for this equation

$$\begin{aligned}
&Q_1=x_0\partial_1+x_1\partial_4,\\
&Q_2=x_1\partial_0+\partial_1,\\
&Q_3=\sqrt{x_0}\partial_1+\sqrt{2u}\partial_u,\\
&Q_4=\partial_1+\ln u\partial_4,\\
&Q_5={x_1}^2\partial_0+2x_1u\partial_1+2u^2\partial_u,\\
&Q_6=\sqrt{2x_0}\partial_1+L(u)\partial_4,
\end{aligned}$$

where $L(u)$ is a solution of the equation $uL''+L'=L^{-1}$.

II. For the equation

$$\left(3\lambda_1+\frac{\lambda_2}{u}\right)u_0+u_{11}=\left(2\lambda_1+\frac{\lambda_2}{u}\right)P_3(u), \tag{6}$$

where $P_3(u)=\lambda_0+\lambda_1u+\lambda_2u^2+\lambda_3u^3$, $\lambda_\mu - const, \quad \mu=\overline{0,4}$, we obtain the operator

$$Q=\partial_0+u\partial_1+P_3(u)\partial_u. \tag{7}$$

Operators of the Q-conditional symmetry can be used to construct and to reduce differential equations and to find their exact solutions. We describe its employment on the example of equation (6); the ansatz for operator (7) has the form

$$x_1-\int\frac{udu}{P_3(u)}=\varphi(\omega), \qquad \omega=x_0-\int\frac{du}{P_3(u)}. \tag{8}$$

This ansatz reduces equation (6) to the ordinary differential equation

$$\ddot\varphi+P_3\left(\dot\varphi\right)=0. \tag{9}$$

The general solution of equation (9) can be written in the parametrical form

$$\omega=-\int\frac{dt}{P_3(t)}, \qquad \varphi=-\int\frac{tdt}{P_3(t)}.$$

Then

$$\left\{\begin{array}{l} x_0=\int\frac{du}{P_3(u)}-\int\frac{dt}{P_3(t)},\\ x_1=\int\frac{udu}{P_3(u)}-\int\frac{tdt}{P_3(t)}.\end{array}\right. \tag{10}$$

We consider the second direction of the investigation of conditional symmetry on the example of the equation

$$u_0+\vec\nabla\left(f(u)\vec\nabla u\right)=0. \tag{11}$$

Theorem 2. *Equation (11) is invariant with respect to the Galilean operators*

$$G_a=x_0\partial_a+M(u)x_a\partial_u \tag{12}$$

with the additional conditions

$$u_0 + \frac{1}{2M(u)}\left(\vec{\nabla}u\right)^2 = 0, \quad M(u) = \frac{u}{2f(u)}.$$

Note. According to the Lie method, the equation (11) is not Galilean-invariant with an arbitrary function $f(u) \neq const$. This fact indicates nonuniversality of the Lie method. Thus, the equation that describes heat processes must satisfy the Galilean principle of relativity.

The example of the third direction of the investigation of conditional symmetry is the following statement.

Theorem 3. *The equation*

$$u_0 + \vec{\nabla}\left(e^u\vec{\nabla}u\right) + \lambda e^{-u} = 0 \tag{13}$$

with the additional condition

$$u_0 + \frac{n}{2}e^u\left(\vec{\nabla}u\right)^2 + \frac{\lambda}{2}e^{-u} = 0 \tag{14}$$

is invariant with respect to the conformal algebra $AC(1,n)$ with the operators

$$\partial_0, \quad \partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad D = x_0\partial_0 + x_a\partial_a + \partial_u, \tag{15}$$

$$\begin{aligned} &J_{0a} = x_a\partial_0 + nw\partial_a, \quad K_0 = 2\lambda wD - \left(w^2 - \frac{\lambda}{n}\vec{x}^2 - e^{2n}\right)\partial_0, \\ &K_a = 2\frac{\lambda}{n}x_aD - \left(w^2 - \frac{\lambda}{n}\vec{x}^2 - e^{2n}\right)\partial_a, \end{aligned} \tag{16}$$

where $w = x_0 + \frac{1}{\lambda}e^u$.

The additional condition (14) extends considerably the symmetry of equation (13), since the Lie symmetry of this equation consists of operators (15). The invariance of a parabolical equation with respect to the conformal algebra is unusual. This can be accounted for the fact that equation (1) has a very wide set of solutions, and this set has invariant subsets with respect to the Galilean and conformal algebras.

2 Nonlocal symmetry

It is well known (see, for example, [1]) that the equation

$$u_0 = \partial_1\left(f(u)u_1\right) \tag{17}$$

reduces to the equation

$$Z_t = \partial_x\left[f^*(z)z_x\right], \quad f^*(z) = z^{-2}f\left(z^{-1}\right) \tag{18}$$

by the chain of the substitutions

1) $x_0 = x_0, \quad x_1 = x_1, \quad u\left(x_0, x_1\right) = \dfrac{\partial v\left(x_0, x_1\right)}{\partial x_1},$

2) $x_0 = t, \quad x_1 = w\left(t, x\right), \quad v\left(x_0, x_1\right) = x,$

3) $t = t, \ x = x, \ \dfrac{\partial w(t,x)}{\partial x} = z(t,x).$

That is to say that these substitutions do not take out the equation from the class of equation (17). We used this fact for finding nonlocal ansatzes which reduce equation (17) to ordinary differential equations, and for constructing nonlocal formulae of generation and superposition of the solutions of this equation at concrete $f(n)$.

Theorem 4. *If $u^{(1)}(x_0, x_1)$ is a solution of the equation*

$$u_0 = \partial_1 \left(u^{-2} u_1 \right), \tag{19}$$

then a new solution $u^{(2)}(x_0, x_1)$ of equation (19) can be found by the formula

$$u^{(2)}(x_0, x_1) = \frac{u^{(1)}(x_0, \tau)}{a x_1 u^{(1)}(x_0, \tau) - x_1 \tau^{-1}}, \tag{20}$$

where a is an arbitrary numerical parameter which can be defined from the following system

$$\begin{cases} \tau_0 = {u^{(1)}}^{-2}(x_0, \tau)(x_0, \tau)\, {\tau_1}^{-2} \tau_{11} + 2a, \\ \tau_1 = \left[a x_1 u^{(1)}(x_0, \tau) + x_1 \tau^{-1} \right]^{-1}. \end{cases} \tag{21}$$

The next example shows efficiency of formulae (20)–(21)

$$u^{(1)}(x_0, x_1) = 1 \longrightarrow \ln \frac{x_1 u^{(2)}(x_0, x_1)}{1 - a x_1 u^{(2)}(x_0, x_1)} + \frac{a x_1 u^{(2)}(x_0, x_1)}{1 - a x_1 u^{(2)}(x_0, x_1)} = a^2 x_0 + \ln x_1.$$

Theorem 5. *If $u^{(1)}(x_0, x_1)$ and $u^{(2)}(x_0, x_1)$ are solutions of equation (19), then the third solution $u^{(3)}(x_0, x_1)$ can be find by the formula*

$$\frac{1}{u^{(3)}(x_0, x_1)} = \frac{1}{u^{(1)}(x_0, \tau^{(1)})} + \frac{1}{u^{(2)}(x_0, \tau^{(2)})}, \tag{22}$$

where $\tau^{(1)}, \tau^{(2)}$ are functional parameters which are defined by the conditions

$$\begin{cases} \tau^{(1)} + \tau^{(2)} = x_1, \\ u^{(1)}\left(x_0, \tau^{(1)}\right) d\tau^{(1)} = u^{(2)}\left(x_0, \tau^{(2)}\right) d\tau^{(2)}, \\ \tau_0^{(k)} = \tau_{11}^{(k)} \left[\tau_1^{(k)} u^{(k)}\left(x_0, \tau^{(k)}\right) \right]^{-2}, \quad k = 1, 2. \end{cases} \tag{23}$$

After the substitutions 1)–3), the equation

$$u_0 = \partial_1 \left(u^{-\frac{2}{3}} u_1 \right) \tag{24}$$

reduces to the equation

$$z_t = \partial_x \left(z^{-\frac{4}{3}} z_x \right). \tag{25}$$

Using the fact that equation (25) has a wider Lie symmetry than equation (24), we can construct the nonlocal ansatz

$$\begin{aligned} &[x_1 + \varphi^1(x_0)] \left[\dot{\varphi}^2(x_0) \right]^{\frac{3}{4}} = \textstyle\int \left[\dot{\varphi}^3(\tau) \right]^{\frac{3}{2}} \varphi^4(\omega) d\tau, \\ &\omega = \varphi^2(x_0) + \varphi^3(\tau), \quad \tau_1 = u, \end{aligned} \tag{26}$$

which reduces equation (24) to the system

$$\begin{cases} \dot\varphi^1 = 0, \qquad \ddot\varphi^2 = \lambda_2 \left(\dot\varphi^2\right)^2, \\ 2(\dot\varphi)^3(\varphi''')^3 - 3\left(\ddot\varphi^3\right)^2 = 2\lambda_1 \left(\dot\varphi^3\right)^4, \\ \left(\varphi^4\right)^{-\frac{4}{3}} \ddot\varphi^4 - \frac{4}{3}\left(\varphi^4\right)^{-\frac{7}{3}} \left(\dot\varphi^4\right)^2 + 3\lambda_1 \left(\varphi^4\right)^{-\frac{1}{3}} + \frac{3}{4}\lambda_2\varphi^4 - \dot\varphi^4 = 0, \end{cases} \tag{27}$$

where $\lambda_i = \mathrm{const}, \quad i = \overline{1,3}$.

Theorem 6. *Any two solutions $u^{(1)}(x_0, x_1)$ and $u^{(2)}(x_0, x_1)$ of the equation*

$$u_0 = \partial_1 \left[\frac{1}{u} f(\ln u)\, u_1\right], \tag{28}$$

where $f(\alpha)$ is an arbitrary smooth even function, are connected by the formula

$$u^{(2)}(x_0, x_1) = \frac{1}{u^{(1)}(x_0, \tau)}, \tag{29}$$

where the functional parameter τ is found from the conditions

$$\tau_0 = \frac{\tau_{11}}{\tau_1} f(\ln \tau_1), \qquad \tau_1 = \frac{1}{u^{(1)}(x_0, \tau)}.$$

References

[1] King I.R., Some non-local transformations between nonlinear diffusion equation, *J. Phys. A: Math. Gen.*, 1990, V.23, 5441–5464.

[2] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993, 436p.