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Integrability of the Perturbed KdV Equation for Convecting Fluids: Symmetry Analysis and Solutions

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Abstract

As an example of how to deal with nonintegrable systems, the nonlinear partial differential equation which describes the evolution of long surface waves in a convecting fluid

$$u_t + \lambda(u_{xxx} + 6uu_x) + 5\beta uu_x + (u_{xxx} + 6uu_x)_x = 0,$$

is fully analyzed, including symmetries (nonclassical and contact transformatons), similarity reductions and the application of the ARS algorithm to the reductions. As a result of the calculations, the Galilean invariance of the equation is shown and all the possible solutions arising from the related ODE through these methods are obtained and classified in terms of the physical parameters.

0. Introduction

Integrable systems are rare in Nature. In instead one encounters often dynamical systems described by Non Linear Partial Differential Equations (NLPDE) which in spite of its wide range of application to physical problems are unfortunately of a nonintegral type. However the definition of integrability may be given (and we have used in this paper a very precise meaning for it) and the interest still lies in dealing with such nonintegrable or almost integrable, or partially integrable Partial Differential Equations (PDE), whose particular exact solutions – in the case they exist – could be of paramount importance in describing such different physical processes as multilayer fluid dynamics, massive transport information through doped optical fibres, gravity–capillarity microwaves, low noise detectors based on nonclassical states of light and about one hundred more physical and even straight technological applications.

This paper is a theoretical attempt in the direction of devising algorithmic procedures dealing with NLPDE which we know to be integrable from the outset. Actually we show in the first part of the paper the importance of the equation in the field of two layer fluid dynamics, but we also show how none of the known procedures based upon Painlevé Tests, Lie Classical Symmetries, Non Classical Blumen and Cole Symmetries and Contact Symmetries gives any clue of how the Equation can be treated to yield some information on the exact solutions that are known experimentally to exist. Then we turn to more advanced – and still algorithmic – methods (with special attention to the Singular Manifold Method) that are able to open different ways to extract information on the exact solutions of this NLPDE and at the same time can be applied to a wide range of other non linear problems.

The paper is divided as follows. First we derive the Equation from first principles mainly based upon the Navier–Stokes equation for fluids with the Rayleigh number above its critical value. In section two the problem of symmetries is analyzed in its various versions. Next we entirely devote section three to Painlevé Analysis and Similarity reductions. It is in section four where we deal with the so–called Conditional Painlevé Property and in section five where we find through the previous analysis a rich class of solutions that are then classified according to the value of the constant parameters. We close with future prospects for further work in this direction.

1. The Perturbed KdV Equation for Convecting Fluids

With the purpose of introducing a good example of a nonintegrable system, this section presents the underlying physical process described by the perturbed Korteveg de Vries (PKdV) equation for convecting fluids. The proposed equation consists basically of a PDE including dissipative and nonlinear terms simultaneously. Our interest about the analytical study of this system was motivated after the work by Garazo and Velarde [1] who found, by means of numerical techniques, soliton–like structures built in the equation. Therefore – although it is not a new result – we found very interesting to follow the derivation of the equation from first principles in order to understand the meaning of the different parameters involved which, as will be seen in the following sections, play a very important role in the integrability of the system and then in the form of its (particular) solutions. For these reasons and relying mainly on the previous work by Aspe and Depassier [2], an step-by-step derivation of the PKdV equation (23) is deduced†. Finally, and for the sake of simplifying its further manipulation, a new rescaling of variables is introduced, leading to the final form (24) [4, 5].

Let us consider in a 2-dimensional geometry (x, z) a fluid layer of thickness d opened to air, subjected to a vertical temperature gradient ∇T and supporting a gravitational field $\vec{g} = -g\vec{k}$. The properties μ (*viscosity*), k (*thermal diffusity*) and α (*coefficient of thermal expansion*) are supposed constant throughout the volume of fluid. Let F represent the prescribed normal heat flux over the free surface, k the thermal conductivity of the fluid and ρ_0, T_0, p_a some reference values for density, temperature and pressure, chosen as the values of these variables at $z = d$ in the static state of the fluid $\rho_s(z), T_s(z), p_s(z)$. From all the characteristic parameters, we can adopt a new system of units given by $[M] = \rho_0 d^3$; $[L] = d$; $[T] = \frac{d^2}{k}$; $[\Theta] = \frac{Fd}{k}$ and then, three dimensionless quantities involved in the problem can be constructed:

$$\sigma = \frac{\mu}{\rho_0 k} \quad \text{(Prandtl Number)},$$

$$R = \frac{\rho_0 g \alpha F d^4}{k \kappa \mu} \quad \text{(Rayleigh Number)},$$

$$G = \frac{g d^3 {\rho_0}^2}{\mu^2} \quad \text{(Galileo Number)}.$$

†Even through the present derivation is initially due to H.Aspe and M.C.Depassier [2], there is a number of references in the literature in which this derivation is presented. See, for instance, [1] or [3].

With these considerations, the dimensionless Navier-Stokes equations governing the problem read (here the subscripts denote derivatives)[6]:

$$u_x + w_z = 0, \tag{1}$$

$$u_t + uu_x + wu_z = -p_x + \sigma\left(u_{xx} + u_{zz}\right), \tag{2}$$

$$w_t + uw_x + ww_z = -p_z + \sigma\left(w_{xx} + w_{zz}\right) + \sigma R\left(T - T_s\right), \tag{3}$$

$$T_t + uT_x + wT_z = T_{xx} + T_{zz}. \tag{4}$$

Furthemore, the fluid is supposed to be bounded from below by a stress-free plane surface maintained at constant temperature T_b, and above by a free surface which is deformed as the fluid moves on. Let $z = 1 + \epsilon^2\nu(x,t)$ denote its position. The boundary (dimensionless) conditions are then [7]:

on $z = 0$:

$$w = u_z = 0, \tag{5}$$

$$T = T_b, \tag{6}$$

on $z = 1 + \epsilon^2\nu(x,t)$:

$$w = \eta_t + u\eta_x, \tag{7}$$

$$p - p_a = \frac{2\sigma}{N^2}\left[w_z + u_x{\eta_x}^2 - \eta_x(u_z + w_x)\right], \tag{8}$$

$$(1 - {\eta_x}^2)(u_z + w_x) + 2\eta_x(w_z - u_x) = 0, \tag{9}$$

$$\vec{n}\cdot\nabla T = -1, \tag{10}$$

where $N = (1 + {\eta_x}^2)^{1/2}$ and $\vec{n} = (-\eta_x, 0, 1)/N$ is the unitary vector normal to the free surface. Therefore, the problem to be solved is described by equations (1) to (10).

Now, we shall introduce a small perturbation to the static solution of the problem. To take account of the slow variation of the waveform, a scale transformation of the independent variables, given by

$$\xi = \epsilon^{\alpha}(x - ct);\ \tau = \epsilon^{\alpha+1}t,$$

is set. As Su and Gardner demonstrated [8], for the final equation to represent a nondissipative system, the value of α must be necessarily $1/2$. Then, redefining the small parameter ($\epsilon^{1/2} \to \epsilon$), a simple analysis of dimensions in the system (1)-(10) leads to the following scaling:

$$u(x,z,t) = \epsilon^2\hat{u}(\xi,z,t); \quad w(x,z,t) = \epsilon^3\hat{w}(\xi,z,\tau); \quad \eta(x,t) = \epsilon^2\hat{\eta}(\xi,\tau);$$
$$p(x,z,t) = p_s(z) + \epsilon^2\hat{p}(\xi,z,\tau); \quad T(x,z,t) = T_s(z) + \epsilon^3\hat{\Theta}(\xi,z,\tau).$$

After the scaling, the nondimensional equations become (dropping the ˆ from all variables):

$$u_\xi + w_z = 0, \tag{11}$$

$$-\epsilon c u_\xi + \epsilon^3 u_\tau + \epsilon^3 uu_\xi + \epsilon^3 wu_z = -\epsilon p_\xi + \epsilon^2\sigma u_{\xi\xi} + \sigma u_{zz}, \tag{12}$$

$$-\epsilon^2 c w_\xi + \epsilon^4 w_\tau + \epsilon^4 uw_\xi + \epsilon^4 ww_z = -p_z + \epsilon^3\sigma w_{\xi\xi} + \epsilon\sigma w_{zz} + \epsilon\sigma R\Theta, \tag{13}$$

$$-\epsilon c\Theta_\xi + \epsilon^3\Theta_\tau + \epsilon^3 u\theta_\xi + \epsilon^3 w\Theta_z - w = \Theta_{zz} + \epsilon^2\Theta_{\xi\xi}, \tag{14}$$

and the scaled boundary conditions:

on $z = 0$:

$$w = u_z = \Theta = 0, \tag{15}$$

on $z = 1 + \epsilon^2\eta(x,t)$:

$$w = -c\eta_\xi + \epsilon^2\eta_\tau + \epsilon^2 u\eta_\xi, \tag{16}$$

$$u_z = -\epsilon^2 w_\xi + \epsilon^6(u_z + \epsilon^2 w_\xi){\eta_\xi}^2 + 2\epsilon^4\eta_\xi(u_\xi - w_z), \tag{17}$$

$$\Theta_z = \epsilon^4\eta_\xi + \epsilon^{-3}[1 - (1 + \epsilon^6{\eta_\xi}^2)^{1/2}], \tag{18}$$

$$p = G\sigma^2\eta + \epsilon^2\frac{\sigma R}{2}\eta^2 + \frac{2\sigma\epsilon}{N^2}(w_z - \epsilon^2\eta_\xi u_z - \epsilon^4\eta_\xi w_\xi + \epsilon^6 u_\xi {\eta_\xi}^2). \tag{19}$$

Regarding ϵ as an ordering parameter, we now look for an asymptotic solution to equations (11)-(19) of the form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots$$

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots$$

$$\Theta = \Theta_0 + \epsilon\Theta_1 + \epsilon^2\Theta_2 + \cdots$$

Besides, as the Rayleigh number R is slightly above its critical value, we shall set $R = R_c + \epsilon^2 R_2$. Under these conditions, every order in ϵ gives a system of inhomogeneous linear equations to be solved sequentially. For the sake of brevity, we here only will comment briefly the results.

In the first two leading orders the velocity component $u(\xi,\tau)$ is independent of z; its solution is then given by $u(\xi,\tau) = f(\xi,\tau) + \epsilon g(\xi,\tau)$, where f and g are arbitrary functions whose evolution equation has to be determined by applying the solubility conditions at the appropriate order. These conditions are systematically found by putting Eq.(12) into the correspondent order in ϵ . At orders ϵ and ϵ^2, it is easily found that the solubility conditions determine the values of the critical phase speed $c^2 = \sigma^2 G$ and the critical Rayleigh number $R_c = 30$, respectively. Taking account of those values and the expressions for all the variables in the preceding orders, the solubility condition of the approximation at order ϵ^3 shows that f obeys the KdV equation

$$f_\tau + \lambda_1 f f_\xi + \lambda_2 f_{\xi\xi\xi} = 0, \tag{20}$$

where

$$\lambda_1 = \frac{3}{2\sigma G}(10 + \sigma G);\ \lambda_2 = \frac{\sigma\sqrt{G}}{2}\left(\frac{1}{3} + \frac{34\sigma}{21}\right).$$

Finally, the evolution equation for g is found by applying the solubility condition at order ϵ^4, which leads to:

$$g_\tau + \lambda_1(fg)_\xi + \lambda_2 g_{\xi\xi\xi} + \frac{\sigma R_2}{15}f_{\xi\xi} + \tilde{\lambda}_3 f_{\xi\xi\xi\xi} + \tilde{\lambda}_4(ff_\xi)_\xi = 0, \tag{21}$$

with the coefficients $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ given by

$$\tilde{\lambda}_3 = \sigma\left(\frac{68\sigma^2 G + 717}{2079}\right); \; \tilde{\lambda}_4 = \frac{8}{\sqrt{G}}.$$

Combining equations (20) and (21) and taking account of their correspondent order in ϵ, we have

$$(f + \epsilon g)_\tau + \lambda_1[ff_\xi + \epsilon(fg)_\xi] + \lambda_2(f + \epsilon g)_{\xi\xi\xi} + \epsilon\left[\frac{\sigma R_2}{15}f_{\xi\xi} + \tilde{\lambda}_3 f_{\xi\xi\xi\xi} + \tilde{\lambda}_4(ff_\xi)_\xi\right] = 0, \quad (22)$$

so that the equation for $u(\xi, \tau)$ at the correct order in ϵ finally yields

$$u_\tau + \lambda_1 u u_\xi + \lambda_2 u_{\xi\xi\xi} + \epsilon\left[\frac{\sigma R_2}{15}u_{\xi\xi} + \tilde{\lambda}_3 u_{\xi\xi\xi\xi} + \tilde{\lambda}_4(uu_\xi)_\xi\right]. \quad (23)$$

Then, the evolution of the surface displacement is governed by a perturbed KdV equation in such a way that the excess of the Rayleigh number above its critical value and the presence of nonlinear terms have a destabilizing effect which is balanced by diffusion.

In order to decrease the number of independent coupling constants in Eq.(23), let $\lambda_3 = \epsilon\tilde{\lambda}_3$, $\lambda_4 = \epsilon\tilde{\lambda}_4$ and $\lambda_5 = \epsilon\frac{\sigma R_2}{15}$. Making use of these new parameters and the rescaling

$$x = \xi + \frac{\lambda_1\lambda_5}{\lambda_4}\tau; \quad t = \lambda_3\tau; \quad u(\xi,\tau) = \frac{6\lambda_3}{\lambda_4}\hat{u}(x,t) - \frac{\lambda_5}{\lambda_4}.$$

Eq.(23) can be transformed into

$$\hat{u}_t + \lambda(\hat{u}_{xxx} + 6\hat{u}\hat{u}_x) + 5\beta\hat{u}\hat{u}_x + (\hat{u}_{xxx} + 6\hat{u}\hat{u}_x)_x = 0,$$

where

$$\lambda = \frac{\lambda_2}{\lambda_3}; \; \beta = \frac{6}{5}\left(\frac{\lambda_1}{\lambda_4} - \frac{\lambda_2}{\lambda_3}\right).$$

Eq.(24) constitutes the final form of the PKdV that shall be analyzed henceforth.

2. Symmetries

As the initial step in the search for solutions of the PKdV equation, in this section an exhaustive analysis of its symmetries is presented, including nonclassical Bluman and Cole symmetries and contact transformations†. As a result of the calculations, the Galilean invariance of the equation is shown and the related similarity reduction is obtained. In the foregoing analysis is found that the Galilean is the only invariance exhibited by the equation, at least until the contact transformations stage. The Lie–Bäcklund symmetries are not considered in the present paper. For further information about this kind of transformations see [9, 10].

† As is argued below, the calculation of the classical (Lie) symmetries is spurious in the present example.

2.1. Nonclassical symmetries

Given a partial differential equation $H(x^i, u, u_{,i}, \cdots) = 0$, where $u = u(x^i)$ and $_{,k} = \dfrac{\partial}{\partial x^k}$, the nonclassical procedure (NCM) to determine its symmetries [9, 10] consists in finding those infinitesimal transformations of the form

$$\tilde{x}^n = \tilde{x}^n(x^i, u), \;\; \tilde{u}^n = \tilde{u}^n(x^i, u), \tag{25}$$

with the generator

$$X = \xi^n(x^i, u)\frac{\partial}{\partial x^n} + \eta(x^i, u)\frac{\partial}{\partial u}, \tag{26}$$

which leaves invariant the equation and verifies the so–called *invariant surface condition* (Eq.(28)). That is, those transformations, for which

$$XH(x^i, u, u_{,i}, \cdots) = 0 \tag{27}$$

and

$$\eta(x^i, u) = \xi^n(x^i, u)u_{,n}, \tag{28}$$

hold. In other words, the nonclassical procedure determines transformations of the form (25) that leave invariant not only the given equation but also certain boundary conditions imposed through Eq.(28).

In the case of the perturbed KdV equation

$$u_{,t} + \lambda(u_{,xxx} + 6uu_{,x}) + 5\beta uu_{,x} + (u_{,xxx} + 6uu_{,x})_{,x} = 0, \tag{29}$$

we deal with two independent variables, so $x^i = x, t$. Then, it is easy to realize that two different cases arise from Eq.(28). Firstly assume $\xi^t \neq 0$, therefore we can set without loss of generality $\xi^t = 1$. A similar simplification can be established for the case $\xi^t = 0$, where $\xi^x = 1$ is supposed.

Introducing these simplifications, Eq.(28) reads:

(Case I. $\xi^t = 1$) $\qquad \eta(x, t, u) = u_{,t} + \xi^x(x, t, u)u_{,x}. \qquad$ (30.1)

(Case II. $\xi^t = 0, \xi^x = 1$) $\qquad \eta(x, t) = u_{,x}. \qquad$ (30.2)

Then, the nonclassical method leads to two different kinds of transformations depending on whether $\xi^t \neq 0$ or $\xi^t = 0$. For the considered equation we have found the expressions of the infinitesimal transformations related to $\xi^t \neq 0$ and the differential equation corresponding to $\xi^t = 0$ which has to be verified by η. As this new equation is more complicated than the initial one (29), the absence of this kind of symmetries is shown through the Singular Manifold Method.

<u>Case I. $\xi^t = 1, \xi^x \equiv \xi$.</u> First of all, we have to find out how do the $u_{,t}$, $u_{,x}$, $u_{,xx}$, $u_{,xxx}$ and $u_{,xxxx}$ transform. For the symmetry condition to make sense, it is obvious that the transformation laws must be of the form

$$\tilde{u}_{,t} = \frac{\partial \tilde{u}}{\partial \tilde{t}}, \; \tilde{u}_{,x} = \frac{\partial \tilde{u}}{\partial \tilde{x}}, \; \tilde{u}_{,xx} = \frac{\partial \tilde{u}_{,x}}{\partial \tilde{x}}, \; \tilde{u}_{,xxx} = \frac{\partial \tilde{u}_{,xx}}{\partial \tilde{x}}, \; \tilde{u}_{,xxxx} = \frac{\partial \tilde{u}_{,xxx}}{\partial \tilde{x}}. \tag{31}$$

In terms of the infinitesimal generator X, these transformation rules mean that the η_t, η_x, η_{xx}, η_{xxx} and η_{xxxx} are the extensions of ξ and η. Therefore, they have to obey

$$\eta_i = \frac{D\eta}{Dx^i} - u_{,x}\frac{D\xi}{Dx^i}, \quad i = x, t, \tag{32.1}$$

$$\eta_{xx} = \frac{D\eta_x}{Dx} - u_{,xx}\frac{D\xi}{Dx}, \tag{32.2}$$

$$\eta_{xxx} = \frac{D\eta_{xx}}{Dx} - u_{,xxx}\frac{D\xi}{Dx}, \tag{32.3}$$

$$\eta_{xxxx} = \frac{D\eta_{xxx}}{Dx} - u_{,xxxx}\frac{D\xi}{Dx}, \tag{32.4}$$

where

$$\frac{D}{Dx^k} = \frac{\partial}{\partial x^k} + u_{,k}\frac{\partial}{\partial u}.$$

Once the extensions are known, we have to use the field equation (29) and the invariant surface condition (30.1) to eliminate both $u_{,t}$ and $u_{,xxxx}$ from the expressions (32). In our case the conditions are

$$u_{,t} = \eta - \xi u_{,x} \tag{33.1}$$

and

$$u_{,xxxx} = -\eta + [\xi - (6\lambda + 5\beta)u]u_{,x} - 6{u_{,x}}^2 - 6uu_{,xx} - \lambda u_{,xxx}. \tag{33.2}$$

After substituting Eq.(33) into the expressions (32), we have to determine the functions ξ and η from the symmetry condition (27), which leads to

$$\eta_t + (5\beta + 6\lambda)(\eta u_x + \eta_x u) + 12\eta_x u_{,x} + 6(\eta u_{,xx} + \eta_{xx}u) + \lambda\eta_{xxx} + \eta_{xxxx} = 0. \tag{34}$$

As both ξ and η do not depend on the derivatives, terms with the same derivative (or a product of derivatives) of u can be equated to zero independently. Setting the coefficient of $u_{,xx}$, $u_{,xxx}$ equal to zero, we find

$$\xi_{,u} = 0, \tag{35}$$

and equating to zero the coefficient of $u_{,x}u_{,xxx}$, we see that

$$\eta_{,uu} = 0. \tag{36}$$

Continuing to equate to zero, successively, the coefficients of $u_{,xxx}, u_{,xx}, \cdots$, and employing Eqs.(35) and (36), we are led to the relations

$$4\eta_{,xu} - 6\xi_{,xx} + \lambda\xi_{,x} = 0, \tag{37}$$

$$6\eta_{,xxu} - 4\xi_{,xxx} + \lambda(3\eta_{,xu} - 3\xi_{,xx}) + 6\eta + 12u\xi_{,x} = 0, \tag{38}$$

$$\begin{aligned} &4\eta_{,xxxu} - \xi_{,xxxx} - 4\xi\xi_{,x} + 3\lambda\eta_{,xxu} - \lambda\xi_{,xxx} + 12\eta_{,x} + \\ &(6\lambda + 5\beta)\eta - \xi_{,t} + 3(6\lambda + 5\beta)u\xi_{,x} + 12u\eta_{,xu} - 6u\xi_{,xx} = 0, \end{aligned} \tag{39}$$

$$\eta_{,xxxx} + 4\eta\xi_{,x} + \lambda\eta_{,xxx} + \eta_{,t} + 6u\eta_{,xx} + (6\lambda + 5\beta)u\eta_{,x} = 0. \tag{40}$$

Now, as Eq.(36) implies that

$$\eta(x, t, u) = \Gamma(x, t)u + \Theta(x, t), \tag{41}$$

all functions in Eq.(38) are independent of u, so that equating to zero independently those terms in which u appears and those with no dependence on u, we find

$$\Gamma(x,t) = -2\xi_{,x}, \tag{42}$$

$$\Theta(x,t) = \frac{1}{3}\left(8\xi_{,xxx} + \frac{9\lambda}{2}\xi_{,xx}\right), \tag{43}$$

and setting Eq.(42) into Eq.(37), we are led to

$$\xi_{,xx} - \frac{\lambda}{14}\xi_{,x} = 0. \tag{44}$$

Now, putting Eqs.(41) and (42) into Eq.(39) gives

$$-9\xi_{,xxxx} - 4\xi\xi_{,x} - 7\lambda\xi_{,xxx} + 12\Theta_{,x} + (6\lambda+5\beta)\Theta - \xi_{,t} = 0, \tag{45}$$

$$(6\lambda+5\beta)\xi_{,x} - 54\xi_{,xx} = 0. \tag{46}$$

Comparing Eqs.(44) and (46) is straightforward to see that both equations are simultaneously verify if

$$7\beta + 3\lambda = 0. \tag{47}$$

On the other hand, putting Eqs.(41) and (42) into Eq.(40) and equating to zero terms in u^2 lead to

$$\xi_{,xxx} + \frac{5\beta+6\lambda}{6}\xi_{,xx} = 0. \tag{48}$$

So that, comparing with Eq.(44), the relations are satisfied if

$$7\beta + 9\lambda = 0. \tag{49}$$

That is, for the three equations (44), (46) and (48) to be verified simultaneously, the parameters λ and β must vanish. This is a very restrictive condition, as we suppose them to be free. For this reason, this solution is meaningless and, in order to satisfy the equations, the condition to impose is

$$\xi_{,x} = 0. \tag{50}$$

Now, from Eqs.(42), (43) and (45) we see that

$$\Gamma(x,t) = 0, \quad \Theta(x,t) = 0, \quad \xi_{,t} = 0. \tag{51}$$

So that, the solution of the system is

$$\xi = \text{const}, \qquad \eta = 0. \tag{52}$$

And then, the symmetry is generated by functions

$$\xi^x = x_0, \quad \xi^t = t_0, \quad \eta = 0. \tag{53}$$

Consequently, for $\xi^t \neq 0$, the nonclassical symmetries of the PKdV equation are reduced to the Galilean transformations generated by

$$X = x_0\frac{\partial}{\partial x} + t_0\frac{\partial}{\partial t}. \tag{54}$$

The Galilean invariance of the equation demonstrated following the nonclassical procedure can be extended to the (classical) Lie case, as the former involves less restrictive conditions than the Lie method. In other words, the number of nonclassical symmetries is reduced enough as to assure that the classical method will not give other symmetries than the Galilean transformations.

Case II. $\xi^t = 0, \xi^x \equiv 1$. Now, from Eq.(30.2) we can set

$$u_{,xxx} = \eta_{,xx} + 2\eta\eta_{,xu} + \eta{\eta_{,u}}^2 + \eta^2\eta_{,uu}, \tag{55.1}$$

$$u_{,xx} = \eta_{,x} + \eta\eta_{,u}. \tag{55.2}$$

If Eqs.(55) are substituted into the expressions (32) so that the x–derivatives are pulled out, and the field equation (29) is used in order to eliminate the t–derivative, after the application of the symmetry condition (35) one finds a single equation that has to be verified by the function η:

$$\begin{aligned}
&\eta_{,t} + (6\lambda + 5\beta)u\eta_{,x} + 6u\eta_{,xx} + \lambda\eta_{,xxx} + \eta_{,xxxx}+\\
&\eta[(6\lambda + 5\beta)\eta + 12\eta_{,x} + 12u\eta_{,xu} + 3\lambda\eta_{,xxu} + 4\eta_{,xxxu}]+\\
&\eta^2(6\eta_{,u} + 6u\eta_{,uu} + 3\lambda\eta_{,xuu} + 6\eta_{,xxuu}) + \eta^3(\lambda\eta_{,uuu} + 4\eta_{,xuuu})+\\
&\eta^4\eta_{,uuuu} + (\eta_{,x} + \eta\eta_{,u})(6\eta + 3\lambda\eta_{,xu} + 6\eta_{,xxu})+\\
&\eta(\eta_{,x} + \eta\eta_{,u})(3\lambda\eta_{,uu} + 12\eta_{,xuu}) + 6\eta_{,uuu}\eta^2(\eta_{,x} + \eta\eta_{,u})+\\
&3\eta_{,uu}({\eta_{,x}}^2 + \eta^2{\eta_{,u}}^2 + 2\eta\eta_{,x}\eta_{,u}) + 4\eta_{,xu}(\eta_{,xx} + 2\eta\eta_{,xu} + \eta{\eta_{,u}}^2 + \eta^2\eta_{,uu})+\\
&4\eta\eta_{,uu}(\eta_{,xx} + 2\eta\eta_{,xu} + \eta{\eta_{,u}}^2 + \eta^2\eta_{,uu}) = 0.
\end{aligned} \tag{56}$$

In any case, the resulting equation is much more complicated than the initial PDE and then trying to obtain the symmetries of Eq.(29) from it has no sense. In any case, there is a procedure that can be extended to any problem which gives full information about the nonclassical symmetries available for a given equation [4]. Estivez and Gordoa proposed a connection between nonclassical symmetries and the Singular Manifold Method (SMM) [11, 12, 13]. Their proposal relies on the idea that all possible solutions obtained through the NCM can be also derived by means of the SMM, and then both SMM and NCM are connected by those solutions.

For a given equation

$$H(x, t, u, u_x, u_t, \ldots) = 0, \tag{57}$$

and according to the generalization of the Painlevé analysis for PDEs due to Weiss, Tabor and Carnevale [14], we first impose the solution of the equation to verify the Painlevé Property (PP), that is

$$u(x, t) = \sum_{j=0}^{\infty} u_j(x, t)[\Phi(x, t)]^{j-\alpha}, \tag{58}$$

where Φ is an arbitrary analytical function and an integer positive number α is called leading index. Now we shall pay attention to those solutions for which the former expansion takes the special form (SMM):

$$u(x, t) = \sum_{j=0}^{\alpha} u_j(x, t)[\Phi(x, t)]^{j-\alpha}, \tag{59}$$

where now Φ is no more arbitrary, but a function called *singular manifold* which is fixed by means of the quantities

$$w = \frac{\Phi_t}{\Phi_x}, \tag{60.1}$$

$$\nu = \frac{\Phi_{xx}}{\Phi_x}, \tag{60.2}$$

$$s = \nu_x - \frac{\nu^2}{2}, \tag{60.3}$$

and the relations among them, due to the compatibility condition $\Phi_{xt} = \Phi_{tx}$:

$$\nu_t = w_x + w\nu, \tag{61.1}$$

$$s_t = w_{xxx} + 2sw_x + s_x w. \tag{61.2}$$

This set constitutes the so–called *singular manifold equations*. The quantities w and s are invariant under homographic transformations but ν is not. This point has a crucial importance in what follows. Note that substitution of (59) into (57) allows us to obtain the expressions for u_j in terms of w, s and ν. The auto-Bäcklund transformation [15] related to fact that u_α is also a solution of (57) is also obtained. Therefore, the u_α can be written as functions of the quantities defined in (60), that is

$$u = u(\nu, w, s). \tag{62}$$

Now, keeping in mind the form of the invariant surface condition (28), the connection between both techniques arises writing out the expressions for u_x and u_t from the solution defined by (62). The dependence of u on ν is very important here, as the Painlevé Property is invariant under homographic transformations. That is, to preserve the invariance of the PP, the singular manifold Φ should appear in (29) only as a function of the homographic invariants w and s, so that the dependence on ν must be pulled out from the expressions too. This condition will allow us to know whether it is possible the existence of nonclassical symmetries with $\xi^t = 0$.

The Painlevé analysis of equation (29) is fully described in [4, 5]. As the particularities of the calculations lay out of the scope of the present paper, we here only detail the results of interest for the purpose of determining the nonclassical symmetries. Once the truncation ansatz (59) is incorporated into Eq.(29), and the quantities involved are expressed in terms of w and s, the solution u takes the form

$$u = u_2 + 2\frac{\Phi_x}{\Phi}\left(\nu - \frac{\beta}{6}\right) - 2\left(\frac{\Phi_x}{\Phi}\right)^2, \tag{63}$$

where u_2 is a particular solution of Eq.(29). The former equation constitutes an auto-Bäcklund transformation between two solutions. The truncated solutions verifies two additional conditions (singular manifold equations)

$$(u_2)_t - w(u_2)_x = 0, \tag{64.1}$$

$$w_x = 0. \tag{64.2}$$

From Eqs.(63) and (64), given a solution of Eq.(29), its derivatives verify the relations

$$u_x = (u_2)_x + 2\frac{\Phi_x}{\Phi}\left[\nu_x + \nu\left(\nu - \frac{\beta}{6}\right)\right] - 2\left(\frac{\Phi_x}{\Phi}\right)^2\left(3\nu - \frac{\beta}{6}\right) + 4\left(\frac{\Phi_x}{\Phi}\right)^3, \tag{65.1}$$

$$u_t = (u_2)_t + 2\frac{\Phi_x}{\Phi}\left[\nu_t + w\nu\left(\nu - \frac{\beta}{6}\right)\right] - 2\left(\frac{\Phi_x}{\Phi}\right)^2\left(3w\nu - \frac{\beta}{6}w\right) + 4w\left(\frac{\Phi_x}{\Phi}\right)^3. \tag{65.2}$$

Now inserting Eqs.(65) into the invariant surface condition (28) and imposing the homographic invariance of solutions, it is straightforward to see that the only possible values for the infinitesimal generators ξ^x, ξ^t and η are

$$\xi^x = -w; \quad \xi^t = 1; \quad \eta = 0. \tag{66}$$

That is, the only nonclassical symmetry exhibited by the truncated solutions is the Galilean invariance with $\xi^t = 1$ and there are no possible symmetries with $\xi^t = 0$. One still may argue that there can be other solutions different from those obtained by means of the SMM, exhibiting symmetries with $\xi^t = 0$, but all the equations studied up to now have shown the direct correspondence between the nonclassical method and the SMM [4].

2.2. Contact symmetries

As is well known [9], a set of transformations in the space of variables $(x^i, u, u_{,i})$,

$$\tilde{x}^n = \tilde{x}^n(x^i, u, u_{,i}),\ \tilde{u} = \tilde{u}(x^i, u, u_{,i}),\ \tilde{u}_{,n} = \tilde{u}_{,n}(x^i, u, u_{,i}), \tag{67}$$

is called a set of *contact transformations* if they satisfy the contact condition that the transformed derivatives $\tilde{u}_{,n}$ are extensions to the derivatives of the transformations of x^n and u, that is, if

$$\tilde{u}_{,n} = \frac{\partial \tilde{u}}{\partial \tilde{x}^n} \tag{68}$$

holds. In terms of the infinitesimal generator

$$X = \xi^n(x^i, u, u_{,i})\frac{\partial}{\partial x^n} + \eta(x^i, u, u_{,i})\frac{\partial}{\partial u} + \eta_n(x^i, u, u_{,i})\frac{\partial}{\partial u_{,n}}, \tag{69}$$

this condition means that the η_i are extensions of ξ^n and η. Therefore, they have to obey

$$\eta_i = \frac{D\eta}{Dx^i} - u_{,n}\frac{D\xi^n}{Dx^i}, \tag{70}$$

where

$$\frac{D}{Dx^i} = \frac{\partial}{\partial x^i} + u_{,n}\frac{\partial}{\partial u} + u_{,ik}\frac{\partial}{\partial u_{,k}}. \tag{71}$$

A contact transformation with the generator X_c will be called *a symmetry* of a partial differential equation $H(x^i, u, u_{,i}, \cdots) = 0$ if

$$X_c H(x^i, u, u_{,i}, \cdots) \equiv 0 \tag{72}$$

holds.

The functions η, ξ^n and η_i are best given in terms of a generating function

$$\Omega(x^i, u, u_i) = u_{,n}\xi^n - \eta, \tag{73}$$

according to

$$\xi^n = \frac{\partial\Omega}{\partial u_{,n}}, \quad \eta = u_i\frac{\partial\Omega}{\partial u_{,i}}, \quad \eta_i = -\left(u_{,i}\frac{\partial\Omega}{\partial u} + \frac{\partial\Omega}{\partial x^i}\right). \tag{74}$$

If Ω is linear in the derivatives $u_{,i}$, then the transformation (67) is an extended point transformation. In order to apply this framework to the PKdV equation (29), we have first to determine the expressions of η_{ik}, η_{ikr} and η_{ikrs} in terms of Ω according to the recurrence rules (74), being understood that now the total derivative operator is given as Eq.(71) shows. Once the prolongations are known, the field equation (29) has to be used to eliminate $u_{,xxxx}$ from the expressions and then determine the function $\Omega(x^i, u, u_{,i})$ from the symmetry condition (72).

In the present example the function Ω depends only on the first derivatives, so that those terms in Eq.(29) depending upon the higher order derivatives can be equated to zero independently. For the sake of brevity we have introduced the notation $u_{,x} \equiv p, u_{,t} \equiv q$, in the foregoing calculation.

Setting the coefficients of $u_{,xt}u_{,xxxt}$ equal to zero, one finds

$$\Omega_{,qq} = 0, \tag{75}$$

so that

$$\Omega(x,t,u,p,q) = \Omega_1(x,t,u,p)q + \Omega_2(x,t,u,p) = 0, \tag{76}$$

and equating to zero the coefficients of $u_{,xx}u_{,xxxt}$ and $u_{,xxxt}$ yields to

$$\Omega_{,pq} = 0, \ (\Omega_1)_{,x} = 0, \ (\Omega_1)_{,u} = 0, \tag{77}$$

which means $\Omega_1 = \Omega_1(t)$. On the other hand, if the coefficient of ${u_{,xxx}}^2$ is equated to zero, we see that

$$\Omega_{,pp} = 0. \tag{78}$$

This condition implies that the function Ω_2 is linear on p and then, the whole function Ω is linear in the derivatives p and q, which characterizes a point transformation.

Thus, if now we set to zero the other coefficients of $u_{,xx}u_{,xxx}, u_{,xxx}, u_{,xx}, \cdots$, it is straightforward to find that the complete expression for Ω is

$$\Omega = t_0 q + x_0 p, \quad t_0, x_0 = \text{const}, \tag{79}$$

and then, taking account of (74), one easily obtains the generating functions of the symmetry

$$\xi^x = x_0, \quad \xi^t = t_0, \quad \eta = 0. \tag{80}$$

That is, all contact symmetries of the PKdV equation are point transformations related to the Galilean invariance of the equation, shown before by the NCM.

2.3. Similarity reductions†

The results of the preceding sections show that, at least until the contact symmetries stage, the symmetries of the PKdV equation are reduced to the galilean transformations. If one allows the transformation to depend upon higher order derivatives (Lie-Bäcklund case) [9, 10], other kind of symmetries may be found, although the nonintegral nature of the equation, demonstrated through the Painlevé analysis [4], indicates that the number of symmetries cannot be very high. Nevertheless, the Galilean invariance leads to a similarity reduction (a *travelling wave* reduction) given by the relation

$$z = x_0 + \nu_0 t, \tag{81}$$

where z is the new independent variable and ν_0 is in principle an arbitrary constant. The similarity reduction transforms the initial partial differential equation into an ordinary differential equation (ODE) given by

$$\nu_0 u' + \lambda(u''' + 6uu') + (u''' + 6uu')' + 5\beta uu' = 0, \tag{82}$$

where the prime means $\dfrac{d}{dz}$ and α_0 is the integration constant. This new equation can be integrated once, yielding the final equation

$$u''' + \lambda u'' + \frac{1}{2}(5\beta + 6\lambda)u^2 + 6uu' + \nu_0 u + \alpha_0 = 0. \tag{83}$$

which will be analyzed henceforth in order to obtain travelling wave solutions of the initial PDE (29).

3. Painlevé analysis of the similarity reduction

The nonintegral behaviour of the PKdV equation (29) implies that the related ordinary differential equation (83) does not have the Painlevé Property for any value of the free parameters λ, β, ν_0, and α_0. Nevertheless, the application of the Conditional Painlevé Property (CPP) method leads to a variety of solutions which can be interpreted as solutions of (29) by means of the transformation (81). In the following sections, this technique is fully described showing that, once one finds the similarity reductions (in this case just one) of a given partial differential equation, the CPP method gives the conditions to impose over the parameters to assure the integrability of those reductions.

Given an ordinary differential equation, it is said to possess the Painlevé Property (PP) if its solutions can be written in the form of a Laurent series expanded about its movable singularities (ARS algorithm [18, 19]), that is

$$u(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{j-\alpha}, \tag{84}$$

where is α a positive integer and $a_j (j \in \mathbf{N})$ are real numbers. That is, for the Painlevé Property to be verified, the solutions of the equation have no movable singularities other than poles.

†By *similarity reduction* must be understood all those ODE derived from a given PDE by means of symmetry transformations. See, for instance, [16, 17].

The coefficients a_j and the leading index α are readily found by substituting the ansatz (84) into the equation. In particular, taking account of the dominant terms in (83) leads to

$$\alpha = 2, \tag{85.1}$$

$$a_0 = -2. \tag{85.2}$$

So that, the equation becomes

$$\sum_{j=0}^{\infty} A_j (z - z_0)^{j-5} + \alpha_0 = 0, \tag{86}$$

where

$$\begin{aligned} A_j = (j-2)(j-3)(j-4)a_j + \lambda(j-3)(j-4)a_{j-1}+ \\ \nu_0 a_{j-3} + \tfrac{6\lambda+5\beta}{2} \textstyle\sum_{n=0}^{j-1} a_{j-n-1}a_n + 6\sum_{n=0}^{j}(n-2)a_{j-n}a_n. \end{aligned} \tag{87}$$

Since z is the independent variable of the expansion, for equation (86) to be satisfied, each order in j must vanish independently. In this sense, (87) can be regarded as a recurrence rule which allows us to determine a_j as a function of $u_0, \cdots, u_{j-1}$ and the parameters. For $j = 5$, the contribution of α_0 must be kept in mind in order to cancel with A_5.

On the other hand, as the coefficient of a_j (with $j \neq 5$) in (87) reads

$$(j+1)(j-4)(j-6)a_j, \tag{88}$$

for $j = -1, 4$ and 6 the equation (83) shows resonances and, if it possesses the Painlevé Property, the coefficients a_4 and a_6 must be arbitrary. The resonance in $j = -1$ can be interpreted taking account of the arbitrariness in choosing z_0. That is, to check whether the equation possesses or not the Painlevé Property, one has to determine the coefficients a_0, a_1, a_2, a_3 and a_5, and prove that the expressions for $j = 4$ and 6 are identically satisfied. Under these conditions, the equation is said to have the PP.

For the sake of brevity, we shall avoid details of the calculations giving only the final expressions for the coefficients, which read:

$$a_0 = -2, \tag{89.1}$$

$$a_1 = -\frac{\beta}{3}, \tag{89.2}$$

$$a_2 = -\frac{\beta}{36}(5\alpha + 4\beta), \tag{89.3}$$

$$6a_3 = \left[\nu_0 - w_0 - \left(\frac{\beta}{6}\right)^3\right], \tag{89.4}$$

$$\begin{aligned} 6a_5 = \alpha_0 - 2a_4(5\lambda + 6\beta) - \tfrac{\beta}{18}(\nu_0 - w_0)(11\lambda + 9\beta)+ \\ \tfrac{1}{6}\left(\tfrac{\beta}{6}\right)^4 (17\lambda + 14\beta) - \tfrac{1}{6^3}\left(\tfrac{\beta}{6}\right)^2 (5\lambda + 4\beta)(6\lambda + 5\beta)(15\lambda + 13\beta), \end{aligned} \tag{89.5}$$

where

$$w_0 = \frac{5\beta}{6}\left(\frac{(6\lambda + 5\beta)}{6}\right)^2. \tag{90}$$

Equating to zero the coefficients F_4 and F_6 gives:

$$(\lambda+\beta)(\nu_0-w_0)=0, \tag{91.1}$$

$$\begin{aligned}&5(3\lambda+5\beta)(2\lambda+3\beta)a_4-\alpha_0(3\lambda+7\beta)+\\&(\nu_0-w_0)^2+(\nu_0-w_0)M(\lambda,\beta)+N(\lambda,\beta)=0,\end{aligned} \tag{91.2}$$

with

$$M(\lambda,\beta)=\frac{\beta}{36}\left[2(3\lambda+7\beta)(11\lambda+9\beta)+\frac{\beta}{6}(3\lambda+\beta)\right], \tag{92.1}$$

$$\begin{aligned}&N(\lambda,\beta)=-\tfrac{1}{6}\left(\tfrac{\beta}{6}\right)^5(3\lambda+2\beta)-\tfrac{1}{36}\left(\tfrac{\beta}{6}\right)^4(3\lambda+7\beta)(17\lambda+14\beta)+\\&\tfrac{1}{216}\left(\tfrac{\beta}{6}\right)^2(3\lambda+7\beta)(5\lambda+4\beta)(6\lambda+5\beta)(15\lambda+13\beta).\end{aligned} \tag{92.2}$$

It is then obvious that conditions (91) are not identically satisfied for all possible values of the free parameters. Therefore, we can conclude stating that equation (83) does not possess the Painlevé Property for any value of λ, β, ν_0 and α_0 as was supposed after the analysis of the initial PDE (29). Nevertheless, these conditions provide full information about the particular solutions of Eq.(83) and then, of Eq.(29). As can be readily seen in the following section, by imposing conditions (91), one finds a classification of solutions according to the number of independent constants appearing in the specific ODE related to each solution. Then, we now seek for these particular values of the free parameters (i.e., the physical conditions), for which equation (83) verifies the Painlevé Condition.

4. Finding solutions: conditional Painlevé property

The above analysis shows that Eq.(83) represents a nonintegral ODE when all the four parameters λ, β, ν_0 and α_0 are free. Furthermore, as the equation is of the third order in the z–derivative, z_0, a_4 and a_6 constitute a complete set of arbitrary constants which should appear in the general solution of the equation if integrable. As the former analysis has shown the nonintegral behaviour of the equation, there are no solutions with three arbitrary constants involving the four free parameters, although by imposing conditions (91), and then, by imposing certain fixed relations among the parameters, it is possible to find out solutions with the specified number of arbitrary constants. Needless to say that these solutions, although general, correspond to very special forms of Eq.(83) and then, the physical processes described by their related solutions of Eq. (29) have a different significance than those described in the fist part of the paper. Through the procedure of imposing conditions over the parameters, we have found the following classification of solutions:

4.1. CASE I

By choosing $\lambda=\beta=0$ in (91.1) and $\nu_0=0$ in (91.2), both conditions are identically satisfied. Equation (83) is then transformed into

$$u'''+6uu'+\alpha_0=0, \tag{93}$$

with α free. Integrating once, this equation reads

$$u''+3u^2+\alpha_0 z+\gamma_0=0, \tag{94}$$

which, by means of the scaling

$$u = -2\omega, \tag{95}$$

is transformed into

$$\omega'' = 6\omega^2 - \alpha_0 z + \gamma_0. \tag{96}$$

The latter equation defines the *First Painlevé Transcendent* (PI) [20]. Then, the solutions of (83) with $\lambda = \beta = 0$ are given in terms of the transcendental function PI, and so are the solutions of the PKdV equation if the parameters are subjected to such a restriction. Note that the condition imposed here over the parameters is the same neglected previously in the symmetry analysis of the initial PDE (29) because of its restrictive significance, since the condition means that the physical constants appearing in (29) have to verify the relation $\lambda_1 = \lambda_2 = 0$.

4.2. CASE II

Now, Eqs.(91) are satisfied by choosing $\nu_0 = w_0$ in (91.1) and imposing the restrictions

$$3\lambda + 5\beta = 0, \tag{97.1}$$

$$-\alpha_0(3\lambda + 7\beta) + N(\lambda, \beta) = 0, \tag{97.2}$$

in (91.2). These conditions fix the values of the phase speed ν_0 and the integration constant α_0 in terms of one of the parameters λ or β. For instance, let the latter to be free, then

$$\nu_0 = \left(\frac{5\beta}{6}\right)^3, \tag{98}$$

and Eq.(83) finally reads:

$$u''' - \frac{5\beta}{3}u'' + \left(\frac{5\beta}{6}\right)^3 u - \frac{5\beta}{2}u^2 + 6uu' + \alpha_0 = 0. \tag{99}$$

As is shown in the following section, this equation can be reduced again in terms of the transcendental function PI.

4.3. CASE III

Lastly, let $\nu_0 = w_0$ in (91.1) and

$$2\lambda + 3\beta = 0, \tag{100.1}$$

$$-\alpha_0(3\lambda + 7\beta) + N(\lambda, \beta) = 0, \tag{100.2}$$

in (91.2). These restrictions lead to

$$\nu_0 = 10\left(\frac{\beta}{3}\right)^3, \tag{101}$$

$$u''' - \frac{3\beta}{2}u'' + \frac{10\beta^3}{27}u - 2\beta u^2 + 6uu' + \alpha_0 = 0, \tag{102}$$

with β free and $\alpha_0 = \alpha_0(\beta)$ given by (100.2). Now, the resulting equation can be integrated in terms of the Weierstrassian elliptic function $\wp(z, g_2, g_3)$. Section 5 presents the details.

These three cases constitute the only relations which lead to general solutions of Eq. (83), although, as was mentioned before, the resulting equations (93), (99) and (102) do not exactly describe the physical problem presented in section 1, but certain restrictions of it. Then, if one tries to obtain analytical solutions of equation (83) with relevance to the motion equation (29), the search for solutions must follow another point of view. In this sense, note that the constant a_4 can be fixed in terms of the parameters by means of Eq.(91.2). Therefore, as a_4 is no more arbitrary, the number of constants in Eq.(83) is reduced to two and then there will be a subset of solutions satisfying the PP but related to equations of the second order.

After these considerations, the next step consists in finding solutions of Eq.(83) taking in mind special values for the parameters involved. Note that the Painlevé analysis has not given us any specific solution but rather provides full information about where we have to seek for them in the parameter space.

5. Particular solutions of the PKdV equation

In order to simplify the form of Eq.(83), consider the following general transformation [21]:

$$u(z) = \mu(z) + M(z)F[(y)] \tag{103}$$

introducing this ansatz into Eq.(83), the following equation is obtained:

$$\begin{aligned}
&M(y')^3 \dddot{F} + 6M^2 y' F\dot{F} + \Big[\lambda M(y')^2 + 3M'(y')^2 + 3My'y''\Big]\ddot{F} + \\
&\Big[6\mu M y' + \lambda(2M'y' + My'') + 3M''y' + 3M'y'' + My'''\Big]\dot{F} + \\
&\Big[\nu_0 M + 2k\mu M + 6(\mu M' + M\mu') + \lambda M'' + M'''\Big] + \\
&(kM^2 + 6MM')F^2 + (\alpha_0 + \nu_0\mu + k\mu^2 + 6\mu\mu' + \lambda\mu'' + \mu''') = 0
\end{aligned} \tag{104}$$

where the y–derivative is represented by a dot and $k = \dfrac{1}{2}(5\beta + 6\lambda)$.

Now, consider the dominant terms ($\dddot{F}$, $F\dot{F}$) in Eq.(104). For their coefficients to be proportional, one has to choose $M = (y')^2$ so that both terms become:

$$(y')^5 \frac{d}{dy}(\ddot{F} + 3F^2). \tag{105}$$

Following the same procedure, consider the next order terms (in $\ddot{F}, F^2$). To simplify the equation, these terms have to become into an expression proportional to $\ddot{F} + 3F^2$, after choosing the appropriate ansatz for the variables in Eq. (104). If one tries to do this, the following expression is found:

$$(y')^4 \left[\left(\lambda + 9\frac{y''}{y'}\right) + \left(k + 12\frac{y''}{y'}\right)F^2\right], \tag{106}$$

and then, the required proportionality between the expressions arises by imposing the coefficients to verify

$$k + 12\frac{y''}{y'} = 3\lambda + 27\frac{y''}{y'}. \tag{107}$$

That is, the transformed variable y has to satisfy the second order linear equation

$$y'' - \frac{\beta}{6} y' = 0. \tag{108}$$

This equation leads to two kinds of solutions depending on the value of β:

$$(\beta = 0) \qquad y = b_0 z, \tag{109.1}$$

$$(\beta \neq 0) \qquad y = b_0 \exp\left(\frac{\beta}{6} z\right) + b_1. \tag{109.2}$$

Now, with these ansatzë and substituting M in terms of y, Eq.(104) becomes

$$\begin{aligned} &\tfrac{d}{dy}(\ddot{F} + 3F^2) + \tfrac{1}{2y'}(2\lambda + 3\beta)(\ddot{F} + 3F^2) + \tfrac{1}{(y')^2}\left(6\mu + \tfrac{5\lambda\beta}{6} + \tfrac{19}{36}\beta^2\right)\dot{F} + \\ &\tfrac{1}{(y')^3}\left[\nu_0 + (6\lambda + 7\beta)\mu + 6\mu' + 4\lambda\left(\tfrac{\beta}{6}\right)^2 + 8\left(\tfrac{\beta}{6}\right)^3\right] F + \\ &\tfrac{1}{(y')^5}(\alpha_0 + \nu_0\mu + k\mu^2 + 6\mu\mu' + \lambda\mu'' + \mu''') = 0. \end{aligned} \tag{110}$$

The integration of the former equation will lead us to particular solutions of the ODE (83). Note that an arbitrary function which will be fixed in each case by convenience and the conditions to impose over Eq.(110) in order to simplify its form must be chosen among the cases analyzed in section 4, which, as was shown there, are the only leading to integrable equations of the third order. The selection of parameters leading to integrable second order equations is also analyzed henceforth.

5.1. CASE $\beta = 0$:

Now, by choosing b_0 in (109.1) and μ, the ansatz (103) reduces to $u \equiv F$ and equation (110) reads:

$$(u'' + 3u^2)' + \lambda(u'' + 3u^2) + \nu_0 + \alpha_0 = 0. \tag{111}$$

As was argued in section 4 (case I), if $\beta = 0$ then necessarily $\lambda = \nu_0 = 0$ in order to verify relations (91). This condition leads straightforward to

$$(u'' + 3u^2)' + \alpha_0 = 0, \tag{112}$$

whose solutions are PI as was shown before.

The solutions of the former equation contain three independent constants, although the conditions imposed over the parameters are severely restrictive. Another way to obtain solutions from equation (83) consists in imposing $\nu_0 = w_0$ in Eq.(91.1), which in the present case implies $\nu_0 = 0$, and choosing $a_4 = a_4(\lambda, \alpha_0)$ in such a way that condition (91.2) was identically satisfied.

Then, in order to linearize (111), let

$$G = u'' + 3u^2, \tag{113}$$

so that G verifies

$$G' + \lambda G + \alpha_0 = 0, \tag{114}$$

whose solution is

$$G(z) = \gamma_0 \exp(-\lambda z) - \frac{\alpha_0}{\lambda}, \tag{115}$$

where γ_0 is an arbitrary constant. The equation for u then becomes:

$$u'' + 3u^2 + \frac{\alpha_0}{\lambda} = \gamma_0 \exp(-\lambda z). \tag{116}$$

This equation is integrable only if $\gamma_0 = 0$ [21], in which case can be reduced in terms of elliptic functions by means of the scaling defined by Eq.(95). Introducing such a transformation, the equation finally reads:

$$\omega'' = 6\omega^2 - \frac{\alpha_0}{\lambda}, \tag{117}$$

whose solution is the Weierstrassian elliptic function $\wp(z - z_0, g_2, g_3)$, where z_0, g_3 are arbitrary constants and $g_2 = \frac{2\alpha}{\lambda}$. Now, introducing the parameters e_1, e_2 and e_3 through

$$e_1 + e_2 + e_3 = 0, \tag{118.1}$$

$$-4(e_1e_2 + e_1e_3 + e_2e_3) = g_2, \tag{118.2}$$

$$4e_1e_2e_3 = g_3, \tag{118.3}$$

and taking account of the usual relations between $\wp$ and the Jacobian elliptic functions [20], the solution of (117) can be transformed into:

$$\omega(z) = e_2 - k^2 {k_0}^2 cn^2[k_0(z - z_0)], \tag{119}$$

where ${k_0}^2 \equiv e_1 - e_3$ is an arbitrary constant and the elliptic modulus k is defined by

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}. \tag{120}$$

Furthermore, the solution (119) can be expressed in terms of hyperbolic functions by taking the limit $k = 1$; according to expressions (118) and (120), this limit implies:

$$e_1 = e_2 = -\frac{e_3}{2} = \frac{{k_0}^2}{3}, \tag{121}$$

in which case, solution (119) becomes

$$\omega(z) = -\frac{2{k_0}^2}{3} + {k_0}^2 \tanh^2[k_0(z - z_0)] \tag{122}$$

and the related (static) solution of the PKdV equation (29) finally reads:

$$u(t, x) = \frac{4{k_0}^2}{3} - 2{k_0}^2 \tanh^2[k_0(x - x_0)], \tag{123}$$

where both x_0, k_0 are arbitrary constants. Note that this solution describes the evolution of waves in the case that the physical parameters in (29) verify $\frac{\lambda_1}{\lambda_4} = \frac{\lambda_2}{\lambda_3}$.

5.2. CASE $\beta \neq 0$:

Now, let $b_0 \equiv 1$, $b_1 \equiv 0$ in (109.2), and $\mu' \equiv 0$. Furthermore, in order to continue the simplification of (110), let the terms in $(\dot{F}, F)$ be proportional to $\dot{F} + \frac{2F}{y}$. The condition to impose is then:

$$\left(\frac{6}{\beta}\right)^2 \left[12\mu + \frac{\beta}{3}\left(5\gamma + \frac{19}{6}\beta\right)\right] = \left(\frac{\beta}{3}\right)^3 \left[v_0 + (6\gamma + 7\beta)\mu + \left(\frac{\beta}{6}\right)^2 \left(4\gamma + \frac{4}{3}\beta\right)\right], \quad (124)$$

which fixes μ in terms of the constants involved in the problem as

$$\mu = \left(\frac{\beta}{6}\right)^2 - \frac{v_0}{6\gamma + 5\beta}. \quad (125)$$

Then, Eq.(110) reads:

$$\begin{aligned} &\frac{d}{dy}(\ddot{F} + 3F^2) + \frac{3}{\beta}(2\gamma + 3\beta)(\ddot{F} + 3F^2) - \\ &\frac{6}{6\gamma + 5\beta}\left(\frac{6}{\beta}\right)^2 \frac{v_0 - w_0}{y^2}\left(\dot{F} + \frac{2F}{y}\right) + \\ &\left(\frac{6}{\beta}\right)^5 \frac{1}{y^5}\left[\alpha_0 + \frac{6\gamma + 5\beta}{2}\left(\frac{\beta}{6}\right)^4 - \frac{{v_0}^2}{2(6\gamma + 5\beta)}\right] = 0. \end{aligned} \quad (126)$$

In order to verify (91) and according to the results of section 4 (cases II and III), let $v_0 = w_0$ and $\alpha_0 = \alpha_0(\beta)$ defined by Eqs.(97) and/or (100), which transforms the latter equation into:

$$\frac{d}{dy}(\ddot{F} + 3F^2) + \frac{3}{\beta}(2\gamma + 3\beta)\frac{1}{y}(\ddot{F} + 3F^2) = 0. \quad (127)$$

Besides, to completely fulfil the condition (91.2), there is an additional relation that has to be verified by parameters γ and β:

$$(3\gamma + 5\beta)(2\gamma + 3\beta) = 0. \quad (128)$$

Then, depending on how is condition (128) verified, two different subcases, leading to very different kinds of solutions, arise when is supposed.

First, assume $3\gamma + 5\beta = 0$ or, in terms of the physical parameters of the PKdV equation, $\frac{\lambda_1}{\lambda_4} = \frac{\lambda_2}{2\lambda_3}$. Then Eq.(127) becomes:

$$\frac{d}{dy}(\ddot{F} + 3F^2) - \frac{1}{y}(\ddot{F} + 3F^2) = 0. \quad (129)$$

This equation constitutes the transformation of (99) through the ansatz (103). Redefining variables in a way similar to (113), the equation can be integrated once giving:

$$\ddot{F} + 3F^2 = \gamma_0 y, \quad (130)$$

whose solutions are expressible in terms of PI as is shown in section 4 (case I).

On the other hand, suppose $2\gamma + 3\beta = 0$ in (128). In such a case, Eq.(127) becomes:

$$\frac{d}{dy}(\ddot{F} + 3F^2) = 0, \quad (131)$$

and then, integrating once, the second order equation to be satisfied by F is

$$\ddot{F} + 3F^2 = \gamma_0, \tag{132}$$

which by means of the scaling $F = -2\omega$ becomes:

$$\ddot{\omega} = 6\omega^2 + \gamma_0, \tag{133}$$

whose solution is given again in terms of the Weierstrassian elliptic function $\wp$ as:

$$\omega(y) = \wp(y - y_0, -2\gamma_0, g_3), \tag{134}$$

where y_0, γ_0 and g_3 are arbitrary constants. Redefining the free parameters as given by Eq.(118), the solution (134) can be expressed in terms of the Jacobian elliptic function $cn(y \mid k)$ as:

$$\omega(y) = e_2 - k^2\gamma^2 cn^2[\gamma(y - y_0)], \tag{135}$$

where $\gamma^2 = e_1 - e_3$ and the elliptic modulus k is defined by (120). A particular solution of (131) can be obtained in terms of hyperbolic functions by taking the limit $k = 1$ in (135). This implies that the constants have to be chosen, for example, as $\gamma_0 = -\frac{2}{3}$ and $g_3 = -\left(\frac{2}{3}\right)^2$. Besides, for the sake of simplicity, let $y_0 = 0$. Then (135) becomes:

$$\omega(y) = -\frac{2}{3} + \tanh^2 y, \tag{136}$$

so that

$$F(y) = \frac{4}{3} - 2\tanh^2 y, \tag{137}$$

and, by means of (103), (109.2) and (125) the expression for $u(z)$ is

$$u(z) = \frac{\beta^2}{54}\left\{\frac{13}{2} + \exp\left(\frac{\beta}{3}z\right)\left[2 - 3\tanh^2\left(\exp\left(\frac{\beta}{6}z\right)\right)\right]\right\}, \tag{138}$$

and the related solution of the PKdV equation reads:

$$u(t,x) = \frac{\beta^2}{54}\left\{\frac{13}{2} + \exp\left(\frac{\beta}{3}(x + v_0 t)\right)\left[2 - 3\tanh^2\left(\exp\left(\frac{\beta}{6}(x + v_0 t)\right)\right)\right]\right\}, \tag{139}$$

where $v_0 = 10\left(\frac{\beta}{3}\right)^3$ and β is free, so that the solution represents the evolution of waves when the physical parameters in (29) are subjected to

$$\frac{\lambda_1}{\lambda_4} = \frac{4}{9}\frac{\lambda_2}{\lambda_3}. \tag{140}$$

As in the case $\beta = 0$, it is possible to obtain solutions with $\beta \neq 0$ by imposing $v_0 = w_0$ and choosing $a_4 = a_4(\lambda, \alpha_0)$ in order to verify (91). In such a case, there is no restriction over the parameters λ and β. Then, Eq.(127) has to be integrated without taking into consideration the condition (128) and the solution obtained will be given in terms of two arbitrary constants. Therefore, let the variable F be redefined as

$$G = \ddot{F} + 3F^2, \tag{141}$$

so that Eq.(127) is transformed into the linear ODE

$$\dot{G} + \frac{3}{\beta}(2\lambda + 3\beta)\frac{G}{y} = 0, \tag{142}$$

and then

$$G(y) = \gamma_0 y^{-\frac{3}{\beta}(2\lambda+3\beta)}, \tag{143}$$

γ_0 being arbitrary. The equation for F becomes:

$$\ddot{F} + 3F^2 = \gamma_0 y^{-\frac{3}{\beta}(2\lambda+3\beta)}, \tag{144}$$

which again is integrable only if $\gamma_0 = 0$ [21]. Introducing the usual scaling $F = -2\omega$, the former equation is transformed into

$$\ddot{\omega} = 6\omega^2, \tag{145}$$

whose solution is the Weierstrassian elliptic function

$$\omega(y) = \wp(y - y_0, 0, g_3), \tag{146}$$

where y_0 and g_3 are the integration constants. In the particular case, in which these constants are chosen as $y_0 = g_3 = 0$, the solution reads:

$$\omega(y) = \frac{1}{(1+y)^2}, \tag{147}$$

and then

$$F(y) = -\frac{2}{(1+y)^2}, \tag{148}$$

so that, by means of (103), (109.2) and (125), the solution of (83) is

$$u(z) = -\frac{\beta}{216}(30\lambda + 19\beta) - \frac{1}{2}\left(\frac{\beta}{6}\right)^2\left[1 + \tanh^2\left(\frac{\beta}{12}z\right)\right]^2, \tag{149}$$

and, finally, the travelling wave solution for (29) reads:

$$u(z) = -\frac{\beta}{216}(30\lambda + 19\beta) - \frac{1}{2}\left(\frac{\beta}{6}\right)^2\left[1 + \tanh^2\left(\frac{\beta}{12}(x + v_0 t)\right)\right]^2, \tag{150}$$

where the phase velocity v_0 is given by

$$v_0 = \frac{5\beta}{6}\left(\frac{6\lambda + 5\beta}{6}\right)^2, \tag{151}$$

and there are no restrictions over the physical parameters.

Apart from the solutions presented throughout this paper, we do know the existence of another one related to the condition $\lambda = -\beta$. This restriction can be used to verify (91.1), together with the appropriate $a_4 = a_4(\lambda, \alpha_0)$ in (91.2) so that there must be a second order ODE related to a such solution. The connection between that second order ODE with (83) is still unknown, although the specific form of the solution can be obtained following the SMM as was shown by Estévez, Gordoa and the authors [4, 5]. The mentioned solution reads:

$$u(t,x) = \frac{1}{6}\left(8{k_0}^2 + \frac{\lambda^2}{6}\right) + \frac{\lambda k_0}{3}\tanh[k_0(x+v_0t+x_0)] - 2{k_0}^2\tanh^2[k_0(x+v_0t+x_0)], \tag{152}$$

where k_0, x_0 are arbitrary constants and the phase velocity v_0 is given by

$$v_0 = \frac{\lambda}{6}\left(4{k_0}^2 - \frac{\lambda^2}{6}\right). \tag{153}$$

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