Lie Algebras of Approximate Symmetries

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Abstract

Properties of approximate symmetries of equations with a small parameter are discussed. It turns out that approximate symmetries form an approximate Lie algebra. A concept of approximate invariants is introduced and the algorithm of their calculating is proposed.

A concept of approximate symmetry of an equation with a small parameter and algorithm of calculating such symmetries were proposed in [1] (see also [3∗–7∗]). Examples of the approximate symmetries show that such symmetries usually do not form a Lie algebra, but form a so-called approximate Lie algebra in sense of definition given in [2].

In this paper, we continue investigation of properties of approximate transformation groups and corresponding Lie algebras. In §1, the concept of the approximate Lie algebra introduced in [2] is discussed. Some properties of approximate symmetries are investigated in §2. The §3 is devoted to approximate invariants and algorithms of their calculating for one- and multiparameter groups.

The following notation is used: \( z = (z_1, \ldots, z_N) \) is an independent variable; \( \varepsilon \) is a small parameter; all functions under consideration are assumed to be locally analytic in their arguments. We write \( F(z, \varepsilon) = o(\varepsilon^p) \) if \( \lim_{\varepsilon \to 0} \frac{F(z, \varepsilon)}{\varepsilon^p} = 0 \) or, equivalently, if \( F(z, \varepsilon) = \varepsilon^{p+1}\varphi(z, \varepsilon) \), where \( \varphi(z, \varepsilon) \) is an analytic function defined in a neighborhood of \( \varepsilon = 0 \) and \( p \) is an arbitrary positive integer. If \( f(z, \varepsilon) - g(z, \varepsilon) = o(\varepsilon^p) \), we write briefly \( f \approx g \).

1 Approximate Lie algebras

Definition 1. A class of first-order differential operators

\[ X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i} \]

such that

\[ \xi^i(z, \varepsilon) \approx \xi^i_0(z) + \varepsilon \xi^i_1(z) + \cdots + \varepsilon^p \xi^i_p(z), \quad i = 1, \ldots, N, \]

with some fixed functions \( \xi^i_0(z), \xi^i_1(z), \ldots, \xi^i_p(z), \quad i = 1, \ldots, N \), is called an approximate operator.
Definition 2. An approximate commutator of the approximate operators $X_1$ and $X_2$ is an approximate operator denoted by $[X_1, X_2]$ and is given by

$$[X_1, X_2] \approx X_1 X_2 - X_2 X_1.$$ 

The approximate commutator satisfies the usual properties, namely:

a) linearity: $[aX_1 + bX_2, X_3] \approx a[X_1, X_3] + b[X_2, X_3], \quad a, b = \text{const},$

b) skew-symmetry: $[X_1, X_2] \approx -[X_2, X_1],$

c) Jacobi identity: $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \approx 0.$

Definition 3. A vector space $L$ of approximate operators is called an approximate Lie algebra of operators if it is closed (in approximation of the given order $p$) under the approximate commutator, i.e., if

$$[X_1, X_2] \in L$$

for any $X_1, X_2 \in L.$ Here the approximate commutator $[X_1, X_2]$ is calculated to the precision indicated.

Example. Consider the approximate (up to $o(\varepsilon)$) operators

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y} + \varepsilon y \frac{\partial}{\partial x}.$$ 

Their linear span is not a Lie algebra in the usual (exact) sense. For instance, the (exact) commutator

$$[X_1, X_2] = \varepsilon^2 \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$$

is not a linear combination of the above operators.

However, these operators span an approximate Lie algebra in the first-order of precision.

2 Algebraic properties of approximate symmetries

Consider a one-parameter approximate group $G_1$ of transformations

$$z'^i \approx f_i(z, a, \varepsilon) = f_i^0(z, a) + \varepsilon f_i^1(z, a) + \cdots + \varepsilon^p f_i^p(z, a) + o(\varepsilon^p), \quad i = 1, \ldots, N, \quad (2.1)$$

in $R^N$ ($a \in R$ is a group parameter) with the generator

$$X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i}. \quad (2.2)$$

Definition 4. The approximate equation

$$F(z, \varepsilon) \approx 0 \quad (2.3)$$

is said to be invariant with respect to the approximate group of transformation (2.1) if

$$F(f(z, a, \varepsilon), \varepsilon) \approx 0 \quad (2.4)$$

for all $z$ satisfying (2.3).
Theorem 1. Let the function \( F(z, \varepsilon) = (F^1(z, \varepsilon), \ldots, F^n(z, \varepsilon)) \), \( n < N \), satisfy the condition
\[
\text{rank } F'(z, 0) \big|_{F(z, 0) = 0} = n,
\]
where \( F'(z, \varepsilon) = \| \partial F^\nu(z, \varepsilon)/\partial z^i \| \) for \( \nu = 1, \ldots, n \) and \( i = 1, \ldots, N \).

Then the equation (2.3) is approximately invariant under the approximate group \( G_1 \) with the generator (2.2) if and only if
\[
XF(z, \varepsilon) \big|_{(2.3)} = o(\varepsilon^p). \quad (2.5)
\]
Equation (2.5) is called the determining equation for approximate symmetries. If the determining equation (2.5) is satisfied, we also say that \( X \) is an approximate symmetry of equation (2.3).

Approximate symmetries satisfy the following properties:

Theorem 2. A set of approximate symmetries of an equation forms an approximate Lie algebra.

Theorem 3. If \( X \) is an approximate symmetry of some equation, then \( \varepsilon X \) is also an approximate symmetry of the same equation.

Let Lie algebra \( L_r \) of approximate symmetries be spanned by the following \( r \) approximate operators
\[
X_{\alpha_0} = X_{\alpha_0,0} + \varepsilon X_{\alpha_0,1} + \ldots + \varepsilon^p X_{\alpha_0,p},
\]
\[
X_{\alpha_1} = \varepsilon X_{\alpha_1,0} + \ldots + \varepsilon^p X_{\alpha_1,p-1},
\]
\[
\ldots
\]
\[
X_{\alpha_p} = \varepsilon^p X_{\alpha_p,0}.
\]
Here \( \alpha_1 = 1, \ldots, r_1 \), \( r_0 + \ldots + r_p = r \), \( X_{\alpha_i,k} = \xi^i_{\alpha_i,k}(z) \partial/\partial z^i \).

Theorem 4. The exact operators \( X_{\alpha_0,0}, X_{\alpha_1,0}, \ldots, X_{\alpha_r,0} \) generate an exact Lie algebra for any \( l = 0, \ldots, p \). For \( l = p \), it is a Lie algebra of exact symmetries of the exact equation \( F(z, 0) = 0 \).

Theorem 5. The approximate operators
\[
Y_{\alpha_0} = X_{\alpha_0,0} + \varepsilon X_{\alpha_0,1} + \ldots + \varepsilon^l X_{\alpha_0,l},
\]
\[
Y_{\alpha_1} = X_{\alpha_1,0} + \varepsilon X_{\alpha_1,1} + \ldots + \varepsilon^l X_{\alpha_1,l},
\]
\[
\ldots
\]
\[
Y_{\alpha_p} = \varepsilon^l X_{\alpha_0,0} + \varepsilon^l X_{\alpha_0,1},
\]
\[
Y_{\alpha_p} = \varepsilon^l X_{\alpha_p,0}
\]
form an approximate (up to \( o(\varepsilon^l) \)) Lie algebra of approximate symmetries.
3 Approximate invariants

Consider a set of the approximate transformations \( \{ T_a \} \):

\[
T_a : z' \approx f^i(z, a, \varepsilon) = f^i_0(z) + \varepsilon f^i_1(z) + \cdots + \varepsilon^p f^i_p(z) + o(\varepsilon^p), \quad i = 1, \ldots, N, \quad (3.1)
\]

in \( \mathbb{R}^N \) generating an approximate \( r \)-parameter group \( G_r \) of transformations with respect to the group parameter \( a \in \mathbb{R}^r \). Let

\[
X_a = \xi^i_a(z, \varepsilon) \frac{\partial}{\partial z^i}
\]

be basic generators of the corresponding approximate Lie algebra.

**Definition 5.** An approximate function \( I(z, \varepsilon) \) is called an approximate invariant of the approximate group \( G_r \) of transformations (3.1), if for each \( z \in \mathbb{R}^N \) and an admissible \( a \in \mathbb{R}^r \)

\[
I(z', \varepsilon) \approx I(z, \varepsilon).
\]

**Theorem 6.** The approximate function \( I(z, \varepsilon) \) is an approximate invariant of the group \( G_r \) with the basic generators (3.2) if and only if the approximate equations

\[
XF(z, \varepsilon) \approx 0 \quad (3.4)
\]

hold.

**Remark.** The equations (3.4) are approximate linear first-order partial differential equations with the coefficients depending on a small parameter.

Consider the case of a one-parameter approximate transformation group with the generator

\[
X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i},
\]

where

\[
\xi^i(z, \varepsilon) \approx \varepsilon^l \left( \xi^i_0(z) + \varepsilon \xi^i_1(z) + \cdots + \varepsilon^{p-l} \xi^i_{p-l}(z) \right) + o(\varepsilon^p), \quad l = 0, \ldots, p, \quad (3.6)
\]

and vector \( \xi_0(z) = (\xi^1_0(z), \ldots, \xi^N_0(z)) \neq 0 \).

**Theorem 7.** Any one-parameter approximate group \( G_1 \) with the generator (3.5), (3.6) has exactly \( N - 1 \) functionally independent (when \( \varepsilon = 0 \)) approximate invariants of the form

\[
I^k(z, \varepsilon) \approx I^k_0(z) + \varepsilon I^k_1(z) + \cdots + \varepsilon^{p-l} I^k_{p-l}(z), \quad k = 1, \ldots, N - 1,
\]

and any approximate invariant of \( G_1 \) can be represented in the form

\[
I(z, \varepsilon) = \varphi_0(I^1, \ldots, I^{N-1}) + \varepsilon \varphi_1(I^1, \ldots, I^{N-1}) + \cdots + \varepsilon^{p-l} \varphi_{p-l}(I^1, \ldots, I^{N-1}) + o(\varepsilon^{p-l}),
\]

where \( \varphi_0, \varphi_1, \ldots, \varphi_p \) are arbitrary functions.
For multiparameter approximate groups, we consider a case when the corresponding approximate Lie algebra is a Lie algebra of approximate symmetries, i.e., it is obtained as a solution of some determining equation and has the form (2.6). Let

\[
\begin{align*}
\text{rank} & \begin{bmatrix}
\xi_{i_0,0}^i(z) \\
\xi_{i_1,0}^i(z) \\
\vdots \\
\xi_{i_l,0}^i(z)
\end{bmatrix} = r^*_1, \\
\end{align*}
\]

Here \(r^*_1 \leq r^*_2 \leq \ldots \leq r^*_p\). Let

\[
s_0 = N - r^*_p, \quad s_1 = N - r^*_{p-1}, \ldots, \quad s_p = N - r^*_0.
\]

**Theorem 8.** In this case, the multiparameter group has \(s_p\) approximate invariants

\[
I^1(z, \varepsilon) \approx I^1_0(z) + \varepsilon I^1_1(z) + \ldots + \varepsilon^p I^1_p(z) \equiv J^1,
\]

\[
I^{s_0}(z, \varepsilon) \approx I^{s_0}_0(z) + \varepsilon I^{s_0}_1(z) + \ldots + \varepsilon^p I^{s_0}_p(z) \equiv J^{s_0},
\]

\[
I^{s_0+1}(z, \varepsilon) \approx \varepsilon \left( I^{s_0+1}_0(z) + \varepsilon I^{s_0+1}_1(z) + \ldots + \varepsilon^p I^{s_0+1}_{p-1}(z) \right) \equiv \varepsilon J^{s_0+1},
\]

\[
I^{s_p}(z, \varepsilon) \approx \varepsilon^p I^{s_p}_0(z) \equiv \varepsilon^p J^{s_p},
\]

with functionally independent functions \(I^k_0(z), k = 1, \ldots, p\) and any approximate invariant of \(G_r\) can be represented in the form

\[
I(z, \varepsilon) \approx \varphi_0(J^1, \ldots, J^{s_0}) + \varepsilon \varphi_1(J^1, \ldots, J^{s_1}) + \ldots + \varepsilon^p \varphi_p(J^1, \ldots, J^{s_p}),
\]

where \(\varphi_0, \varphi_1, \ldots, \varphi_p\) are arbitrary functions.
References


[7] * Euler N. and Euler M., Symmetry properties of the approximations of multidimensional generali-

†References [3∗–7∗] were added by editor.