Nonlinear Mathematical Physics 1996, V.3, N 1–2, 90–95.

Parasupersymmetries and Non-Lie Constants of Motion for Two-Particle Equations

Violeta TRETYNYK

Institute of Mathematics of the National Ukrainian Academy of Sciences,
3 Tereshchenkivska Street, Kyv 4, Ukrana

Abstract

We search for hidden symmetries of two-particle equations with oscillator-equivalent potential proposed by Moshinsky with collaborators. We proved that these equations admit hidden symmetries and parasupersymmetries which enable easily to find the Hamiltonian spectra using algebraic methods.

1 Lie Symmetries

Let us consider the two-particle equations of Moshinsky et al. [1–4] in c.m. frame

$$L_1\psi = \Big\{(\vec{\alpha}_1 - \vec{\alpha}_2)\Big(\mathbf{p} - i\frac{\omega}{2}\mathbf{x}\beta_1\beta_2\Big) + m(\beta_1 + \beta_2) - E'\Big\}\psi = 0, \tag{1.1}$$

and [4]

$$L_2\psi = \Big\{(\vec{\alpha}_1 - \vec{\alpha}_2)\Big(\mathbf{p} - i\frac{\omega}{2}\mathbf{x}\beta_1\beta_2\gamma_{51}\gamma_{52}\Big) + m(\beta_1 + \beta_2) - E'\Big\}\psi = 0, \tag{1.2}$$

where $\vec{\alpha}_1,\ \beta_1,\ \gamma_{51}$ and $\vec{\alpha}_2,\ \beta_2,\ \gamma_{52}$ are commuting sets of 16×16 matrices for the first and second particles, correspondingly.

Our interest in equations (1.1), (1.2) is connected with their parasupersymmetric nature. Making the similarity transformation

$$\psi \to \psi' = \beta_2\psi, \quad L_\nu \to L'_\nu = \beta_2 L_\nu \beta_2, \ \ \nu = 1, 2,$$

we reduce (1.1), (1.2) to the form

$$L'_1\psi' \equiv \Big\{[\beta_0, \beta_a]\Big(p_a + \frac{i\omega x_a\eta}{2}\Big) + \beta_0 m - E\Big\}\psi' = 0, \tag{1.3}$$

$$L'_2\psi' \equiv \Big\{[\beta_0, \beta_a]\Big(p_a - \frac{i\omega x_a\xi}{2}\Big) + \beta_0 m - E\Big\}\psi' = 0, \tag{1.4}$$

where

$$\eta = 1 - 2\beta_0^2, \qquad \xi = (1 - 2\beta_0^2)(1 - 2\beta_5^2), \qquad E = \frac{1}{2}E',$$

$$\begin{aligned}
&\gamma_0^{(i)} = \beta_i, \gamma_a^{(i)} = \beta_i \alpha_{ai}, \quad \gamma_5^{(i)} = \gamma_{5i}, \quad i = 1, 2, \\
&\beta_\mu = \tfrac{1}{2}(\gamma_\mu^{(1)} + \gamma_\mu^{(2)}), \qquad \mu = 0, 1, 2, 3, 5.
\end{aligned} \tag{1.5}$$

The equations (1.3), (1.4) are more convenient for symmetry analysis than (1.1), (1.2). Indeed, the matrices β_μ satisfy the Kemmer–Duffin–Petiau (KDP) algebra

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = g_{\mu\nu} \beta_\lambda + g_{\nu\lambda} \beta_\mu \tag{1.6}$$

which enables to use the known results [5] connected with complete sets of irreducible KDP matrices.

Using the classical Lie algorithm (see, e.g., references [6, 7]), it is possible to prove that equations (1.3), (1.4) are invariant under a 6-parametrical Lie group, whose generators are

$$\begin{aligned}
&J_a = \varepsilon_{abc}(x_b p_c + i\beta_b \beta_c), \\
&Q_1 = (1 + \gamma_\mu^{(1)} \gamma^{(2)\mu})(1 + 2\gamma_\mu^{(1)} \gamma^{(2)\mu}), \\
&Q_2 = -(3 + 2\gamma_\mu^{(1)} \gamma^{(2)\mu})\gamma_\mu^{(1)} \gamma^{(2)\mu}, \ \ Q_3 = 1 - Q_1 - Q_2,
\end{aligned} \tag{1.7}$$

where covariant summation is imposed over repeated indices $\mu = 0, 1, 2, 3$.

The operators J_a are generators of the rotations group $O(3)$. As to Q_1, Q_2 and Q_3, these symmetries exist due to the well-known fact that a 16×16-dimensional representation of the KDP algebra is reducible and includes 10×10, 5×5 and 1×1 (trivial) irreducible representations. It means that the equation (1.3), (or (1.4)) can be reduced to three noncoupled subsystems for ten-, five- and one-component functions.

Thus, denoting $\psi = \text{column}(\psi_{(10)}, \psi_{(5)}, \psi_{(1)})$ where $\psi_{(10)}, \psi_{(5)}, \psi_{(1)}$ are ten-, five- and one-component functions, we obtain from (1.3)

$$(H_1 - E)\psi_{(10)} \equiv \Big\{[\beta_0^{(10)}, \beta_a^{(10)}]\Big(p_a + \frac{i\omega x_a \eta^{(10)}}{2}\Big) + \beta_0^{(10)} m - E\Big\}\psi_{(10)} = 0, \tag{1.8}$$

$$(H_0 - E)\psi_{(5)} \equiv \Big\{[\beta_0^{(5)}, \beta_a^{(5)}]\Big(p_a + \frac{i\omega x_a \eta^{(5)}}{2}\Big) + \beta_0^{(5)} m - E\Big\}\psi_{(5)} = 0, \tag{1.9}$$

$$E\psi_{(1)} = 0. \tag{1.10}$$

As to (1.4), it reduces to the ten-component equation

$$(H_1 - E)\psi_{(10)} \equiv \Big\{[\beta_0^{(10)}, \beta_a^{(10)}]\Big(p_a - \frac{i\omega x_a \xi^{(10)}}{2}\Big) + \beta_0^{(10)} m - E\Big\}\psi_{(10)} = 0 \tag{1.11}$$

and to equations (1.9), (1.10) for five- and one-component functions.

2 Non-Lie Symmetries

Besides invariance with respect to the group $O(3)$, equations (1.3), (1.4) are invariant under a space-inversion transformation

$$\psi'(\mathbf{x}) \to \eta\psi'(-\mathbf{x}),$$

where η is the matrix defined in (1.5). From this it follows that these equations admit a non-Lie symmetry which we have called a Dirac-type constant of motion. Indeed, it is not difficult to verify by direct calculation that the operator [8]

$$Q_4 = \eta(2(\mathbf{S}\cdot\mathbf{J})^2 - 2\mathbf{S}\cdot\mathbf{J} - \mathbf{J}^2), \tag{2.1}$$

where $\mathbf{S} = i\beta\times\beta, \mathbf{J}$ is a vector, whose components J_a are defined in (1.7), commutes with L' of (1.3) and (1.4).

Using (1.6), we find the following cubic relation for H of (1.3)

$$H^3 = H(Q_5 + m^2 - \omega) + \omega Q_6, \tag{2.2}$$

where

$$Q_5 = \mathbf{p}^2 + \frac{1}{4}\omega^2\mathbf{x}^2 + \frac{1}{2}\omega\eta, \tag{2.3}$$

$$Q_6 = m[\beta_5,\beta_a]L_a - i\varepsilon_{abc}\beta_a[\beta_5,\beta_b]\Big(p_c + \frac{i}{2}\omega x_c\eta\Big), \quad \vec{L} = \vec{x}\times\vec{p}. \tag{2.4}$$

It is easy to show that Q_5 commutes with L'_1 and so is a symmetry of (1.3). It follows from (2.2) that Q_6 commutes with H and so is one more symmetry of (1.3).

Repeating the above reasoning for equation (1.4), we recognize that in addition to (2.1) there exists just one more symmetry

$$Q_7 = \mathbf{p}^2 + \frac{1}{4}\omega^2\mathbf{x}^2 - \frac{1}{2}\omega\hat{\eta}, \tag{2.5}$$

where

$$\hat{\eta} = 3 - 2\beta_0^2 - 4\beta_5^2(1-\beta_0^2). \tag{2.6}$$

It is not difficult to verify that symmetries (2.1)–(2.5) commute with generators (1.7).

Thus in addition to Lie symmetries (1.7), equation (1.3) has three non-Lie constants of motion (2.1), (2.3), (2.4), moreover, these operators form a basis of the 9-dimensional Lie algebra satysfying the following relations

$$[J_a, J_b] = i\varepsilon_{abc}J_c, \;\; [Q_a, J_b] = [Q_a, Q_b] = 0, \;\; A, B = 1, 2, \ldots 6. \tag{2.7}$$

Lie and non-Lie symmetries of equation (1.4) are given in (1.7) and (2.1), (2.5); they satisfy relations (2.7) with $A, B = 1, 2, 3, 4, 7$.

3 Hidden Parasupersymmetries (PSS)

Let us investigate hidden symmetries of (1.3), (1.4) which appear to have structures typical of parasupersymmetric quantum mechanics (PPSQM).

The Hamiltonian H of (1.3) can be expressed in the form

$$H = \hat{Q}_1 + \beta_0 m \tag{3.1}$$

where $\hat{Q}_1$ is a parasupercharge

$$\hat{Q}_1 = [\beta_0, \beta_a]\Big(p_a + \frac{i\omega x_a \eta}{2}\Big).$$

Indeed, $\hat{Q}_1$ and $\hat{Q}_2 = i\eta\hat{Q}_1$ together with $H_{PSS} = Q_5 + \omega(1 - \beta_5^2)$ (where Q_5 is given in (2.3)) satisfy relations

$$[H_{PSS}, \hat{Q}_A] = 0, \quad A, B, C = 1, 2,$$
$$[\hat{Q}_A, [\hat{Q}_B, \hat{Q}_C]] = 4(\delta_{AB}\hat{Q}_C - \delta_{AC}\hat{Q}_B)H_{PSS},$$

which characterize the algebra of PPSQM [10].

Equation (1.4) also possesses hidden parasupersymmetries. The corresponding Hamiltonian admits the representation (3.1), moreover, parasupercharges and parasuperHamiltonian have the form

$$\hat{Q}_1 = [\beta_0, \beta_a]\Big(p_a - \frac{i\omega x_a \xi}{2}\Big), \;\; \hat{Q}_2 = i[\beta_0, \hat{Q}_1], \;\; H_{PSS} = \frac{1}{4}Q_7,$$

where ξ and Q_7 are given in (1.5) and (2.5).

We note that the Hamiltonian H of (1.4) also is a parasupercharge, inasmuch as

$$H^3 = (Q_7 + m^2)H, \;\; [H, Q_7 + m^2] = 0, \;\; H\xi + \xi H = 0. \tag{3.2}$$

It follows from (3.2) that the operators

$$\hat{Q}_1 = H, \;\; \hat{Q}_2 = i\xi H, \;\; H_{PSS} = Q_7 + m^2$$

satisfy the algebra of the Beckers–Debergh version of PPSQM [10].

We note that all the results of this section are valid for the reduced equations (1.8)–(1.11), inasmuch as they are based on relations (1.6) satisfied by matrices β_μ and $\beta_\mu^{(10)}$, $\beta_\mu^{(5)}$ as well.

4 Hamiltonian Eigenvalues for Parastates and Ortostates

In this section we use symmetries and hidden parasupersymmetries of (1.3), (1.4) to find eigenvalues of the Hamiltonians by purely algebraic methods without solving corresponding equations.

First, we consider the simplest nontrivial subsystem, i.e., (1.9). The equation (1.9) describes spin zero [9] or parastates.

To find possible eigenvalues of E, we use the fact that for 5×5 KDP matrices $\beta_5^{(5)} \equiv 0$, and relation (2.2) reduces to the form

$$H_0^3 = H_0(Q_5 + m^2 - \omega). \tag{4.1}$$

Inasmuch as $[H_0, Q_5] = 0$, the relation (4.1) leads to the corresponding relation for eigenvalues E of H_0 and q of Q_5

$$E(E^2 - q - m^2 + \omega) = 0.$$

In accodance with (2.3),

$$q = (2N + 3 + e)\frac{\omega}{2},$$

where $N = 2n + j$, $n = 0, 1, 2, ...,$ $j = 0, 1, ..., N$; and $e = \pm 1$ are eigenvalues of η. Thus,

$$E = \mu\sqrt{(2N + 1 + e)\frac{\omega}{2} + m^2},$$

where $\mu = \pm 1$, or

$$E = 0.$$

Inasmuch as the matrix η does not commute with H_0 , the values of μ and e are not independent. Using the Foldy–Woythoyzen transformation, it is possible to show that $e\mu = -\mu$ and so nonzero values of E are

$$E = \pm\sqrt{N\omega + m^2}. \tag{4.2}$$

Thus, we find algebraically the known eigenvalues [4] of E for parastates.

Consider the equation (1.8) describing spin-one states. To find eigenvalues of H_1, we use relation (2.2). After some algebraic transaformation we can obtain

$$H_1^2(H_1^2 - Q_5 - m^2)(H_1^2 - Q_5 - m^2 + \omega) = \frac{m^2\omega^2}{2}(\mathbf{J}^2 + Q_4). \tag{4.3}$$

Let us replace in (4.3) commuting operators H_1^2, Q_5, $\mathbf{J}^2$ and Q_4 by their eigenvalues E^2, $(2N + 1 + e)\omega/2$, $j(j + 1)$ and $\nu j(j + 1)$, where $j = 0, 1, 2, \ldots,$ $\nu = \pm 1$. As a result we obtain

$$E^2\Big(E^2 - m^2 - (2N + 1 + e)\frac{\omega}{2}\Big)\Big(E^2 - m^2 - (2N + 3 + e)\frac{\omega}{2}\Big) =$$
$$\frac{m^2\omega^2}{2}j(j + 1)(\nu + 1). \tag{4.4}$$

For $\nu = -1$ we have three possibilities

$$E = 0, \tag{4.5}$$

$$E = \mu\sqrt{m^2 + (2N + 1 + e)\frac{\omega}{2}}, \quad \mu = \pm 1, \tag{4.6}$$

$$E = \mu\sqrt{m^2 + (2N + 3 + e)\frac{\omega}{2}}. \tag{4.7}$$

Like the case of parastates we conclude that values of μ and e are not independent. Using non-relativistic approximation it is possible to show that in (4.6) $\mu e = \mu$ and in (4.7) $\mu e = -\mu$, and so nonzero energies are defined by the relation

$$E = \pm\sqrt{m^2 + (N + 1)\omega}. \tag{4.8}$$

For $\nu = 1$ (4.4) reduces to the third-order algebraic equation for E^2

$$E^2(E^2 - m^2 - (N + 1)\omega)\ (E^2 - m^2 - (N + 2)\omega) = m^2\omega^2 j(j + 1). \tag{4.9}$$

Formulae (4.2), (4.8), (4.9) are in good accordance with the results of Moshinsky et al. [1–4]. Using hidden symmetries of (1.3), (1.4), we obtain these results in a straightforward and easy way. The eigenvalues problem (1.4) also can be solved algebraically using (3.2). Replacing in (3.2) operators H, Q_7 by their eigenvalues E^2 and $q = (2N + 3 + -e)\omega/2$ (where e are eigenvalues of the matrix $\hat{\eta}$ (2.6); moreover, $e\mu = -\mu$, where μ is the energy sign), we come to the relations

$$E = 0 \ \text{ or } \quad E^2 = m^2 + (N + 2)\omega.$$

This formula was already obtained [3] by solving equation (1.2) analytically.

I would like to thank Professor A.G. Nikitin for the proposing of the problem and helpful discussions.

References

[1] Moshinsky M. and Loyola G., in Workshop on Harmonic Oscillators, Proceeding of a Conference, Univ. of Maryland, March 25–28, 1992, NASA Conference Publications 3197, 1993, 405–421.

[2] Moshinsky M.,Loyola G. and Villegas C., *J. Math. Phys.*, 1991, V.32, 373.

[3] Moshinsky M.,Loyola G., Szcezepaniak A., Villegas C. and Aquino V., in Relativistic Aspects of Nuclear Physics, Proc. of the Rio de Janeiro International Workshop, World Scientific, Singapore, 1990, 271-307.

[4] Del Sol Mesa A. and Moshinsky M., *J.Phys.A*, 1994 V.27, 4685.

[5] Kemmer N., *Proc. Camb. Phys. Soc.*, 1943, V.39, 189.

[6] Fushchych W.I. and Nikitin A.G., Symmetries of Equations of Quantum Mechanics, Allerton Press, New York, 1994.

[7] Miller W., Symmetry and Separating of Variables, Addison-Wesley, 1977.

[8] Nikitin A.G. and Fushchych W.I., *Teor. Mat. Fiz.*, 1991,V.88, 406.

[9] Beckers J., Debergh N. and Nikitin A.G., *J. Math. Phys.*, 1992, V.33, 3387.

[10] Beckers J. and Debergh N., *Nucl. Phys. B*, 1990, V.340, 767.