Nonlinear Mathematical Physics 1996, V.3, N 1–2, 147–151.

Symmetry Approach in Boundary Value Problems

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Abstract

The problem of construction of boundary conditions for nonlinear equations compatible with their higher symmetries is considered. Boundary conditions for the sine-Gordon, Zhiber–Shabat and KdV equations are discussed. New examples are found for the JS equation.

1 Introduction

The subject of applications of classical Lie symmetries to boundary value problems is well studied (see the monograph [1]). In contrast, the question of involving higher symmetries in the same problem has received much less attention, unlike, say, the Cauchy problem. However, one should stress that nowadays the higher symmetries' approach becomes the basis of the modern integrability theory [2]. A number of attempts to apply the inverse scattering method (ISM) to an initial boundary value problem have been undertaken. It turned out that if both initial data and boundary value are chosen arbitrary, then the ISM essentially loses its efficiency. On the other, the investigation by E. Sklyanin [3] based on the R-matrix approach demonstrated that there is a kind of boundary conditions, compatible completely with integrability. The analytical aspects of problems of such problems were studied in [4], [5]. After [6], it becomes clear that boundary-value problems found can be efficiently investigated with the help of the Bäcklund transformation.

Below we will discuss a higher symmetry test proposed in [7], [8] to verify whether the given boundary condition is compatible with the integrability property of an equation. It is worthwhile to note that all known classes of boundary conditions compatible with integrability occur to pass this symmetry test. Boundary conditions involving explicit time dependence for the Toda lattice compatible with higher symmetries have recently been studied in [9]. It was established there that finite-dimensional systems obtained from the Toda lattice by imposing boundary conditions consistent with symmetries at both ends were nothing else but Painlevé-type equations.

Let us consider an evolution-type equation

$$u_t = f(u, u_1, u_2, ..., u_n) \tag{1}$$

and a boundary condition of the form

$$p(u, u_1, u_2, ..., u_k)|_{x=0} = 0, \tag{2}$$

imposed at the point $x = 0$. Here u_i stands for the partial derivative of the order i with respect to the variable x. Suppose that the equation given possesses a higher symmetry

$$u_\tau = g(u, u_1, ..., u_m). \tag{3}$$

We call the problem (1)–(2) compatible with symmetry (3) if for any initial data prescribed at the point $t = 0$, a common solution to equations (1), (3) exists satisfying the boundary condition (2). Let us explain what we mean more exactly. Evidently, one can differentiate the constraint (2) with respect to the variables t and τ only (but not with respect to x). For instance, it follows from (2) that

$$\sum_{i=0}^{n} \frac{\partial p}{\partial u_i}(u_i)_\tau = 0, \tag{4}$$

where one should replace τ-derivatives by means of equation (3). The boundary value problem (1)–(2) be compatible with the symmetry (3) if equation (4) holds identically by means of the condition (2) and its consequences obtained by differentiation with respect to t.

To formulate an efficient criterion of compatibility of a boundary value problem with a symmetry, it is necessary to introduce some new set of dynamical variables consisting of the vector $v = (u, u_1, u_2, ...u_{n-1})$ and its t-derivatives: v_t, v_{tt}, ... Passing to this set of variables allows one to exclude the dependence on the variable x. In terms of these variables, the symmetry (3) and the constraint (2) take the form

$$v_\tau = G\left(v, v_t, v_{tt}, ...\frac{\partial^{m_1} v}{\partial t^{m_1}}\right), \tag{5}$$

$$P\left(v, \frac{\partial v}{\partial t}, ..., \frac{\partial^{k_1} v}{\partial t^{k_1}}\right) = 0. \tag{6}$$

The following criterion of compatibility was established in ([8]).

Theorem. *The boundary-value problem (1)-(2) is compatible with the symmetry (3) if and only if the differential connection (6) is consistent with system (5).*

We call the boundary condition (2) compatible with the integrability property of equation (1), if the problem (1)–(2) is compatible with infinite series of linearly independent higher-order symmetries.

The problem of classification of integrable boundary conditions is solved completely for the Burgers equation (see [8])

$$u_t = u_2 + 2\,u\,u_1. \tag{7}$$

Theorem. *If the boundary condition $p(u, u_1)|_{x=0} = 0$ is compatible at least with one higher symmetry of the Burgers equation (7), then it is compatible with all even-order homogeneous symmetries and is of the form $c_1(u_1 + u^2) + c_2\,u + c_3 = 0$.*

In the Burgers case, boundary conditions of the general form (2) can also be described completely with the help of the "recursion operator for the boundary conditions" $L = \frac{\partial}{\partial x} + u$, which acts on the set of integrable boundary conditions (see [10]). For instance, the boundary condition $L(c_1(u_1+u^2)+c_2\,u+c_3) = c_1(u_2+3uu_1+u^3)+c_2(u_1+u^2)+c_3\,u = 0$ is also integrable.

Let us describe boundary-value problems of the form

$$a(u, u_x)|_{x=0} = 0, \tag{8}$$

$$u_{tt} - u_{xx} + \sin u = 0, \tag{9}$$

for the sine-Gordon equation compatible with the third-order symmetry.

As is shown in [11], the complete algebra of higher symmetries for the equation (9) i.e. $u_{\xi\eta} = \sin u$, where $2\xi = x+t$, $2\eta = x-t$, splits into the direct sum of two algebras consisting of symmetries of the equations $u_\tau = u_{\xi\xi\xi} + u_\xi^3/2$, $u_\tau = u_{\eta\eta\eta} + u_\eta^3/2$, correspondingly, which are nothing else but the potentiated MKdV equation. Particularly, the following flow commutes with the sine-Gordon equation

$$u_\tau = c_1(u_{\xi\xi\xi} + u_\xi^3/2) + c_2(u_{\eta\eta\eta} + u_\eta^3/2). \tag{10}$$

The symmetry (10) is not compatible with any boundary condition of the form (8) unless $c_1 = -c_2$, under this constraint the equation (8) is of one of the forms

$$u = \text{ const}, \qquad v = c_1 \cos(u/2) + c_2 \sin(u/2). \tag{11}$$

Note that the list of boundary conditions (11) coincides with that found by A. Zamolodchikov within the framework of the R-matrix approach [12]. The latter of (11) was studied in particular cases earlier in [3] and [5]. The compatibility of the former of (11) with the usual version of ISM was declared earlier in [5]. But the statement was based on a mistake (see [13]). Our requirement of consistency is weaker than that is used in [5]. Applications of these and similar problems for the sine-Gordon equation and the affine Toda lattice in quantum field theory are studied in [14] and [15].

According to the theorem above, one reduces the problem of finding integrable boundary conditions to the problem of looking for differential connections admissible by the following system of equations equivalent to (10) with $c_1 = -c_2$ and $v = u_x$:

$$\begin{aligned} u_\tau &= 8u_{ttt} + 6u_t \cos u + 3v^2 u_t + u_t^3, \\ v_\tau &= 8v_{ttt} + 6v_t \cos u + 6u_{tt} v u_t + 3v^2 v_t + 3u_t^2 v_t. \end{aligned} \tag{12}$$

One can prove that the boundary conditions (11) are compatible with a rather large subclass of the sine-Gordon equation such that

$$u_\tau = \phi(u, u_1, ...u_{k_1}) - \phi(u, \bar{u}_1, ...\bar{u}_{k_1}), \tag{13}$$

where $u_j = \partial^j u/\partial \xi^j$, $\bar{u}_j = \partial^j u/\partial \eta^j$, and the equation $u_\tau = \phi_i(u, u_1, ...u_{k_i})$, $i = 1, 2$ is a symmetry of the equation $u_\tau = u_{\xi\xi\xi} + u_\xi^3/2$.

Another well-known integrable equation of the hyperbolic type

$$u_{tt} - u_{xx} = \exp(u) + \exp(-2u) \tag{14}$$

has applications in geometry of surfaces. For the first time, it was found by Tzitzeica [16]. The presence of higher symmetries for this equation has been established by A. Zhiber and A. Shabat [11]. The simplest higher symmetry of this equation is of the fifth order

$$u_\tau = u_{\xi\xi\xi\xi\xi} + 5(u_{\xi\xi}u_{\xi\xi\xi} - u_\xi^2 u_{\xi\xi\xi} - u_\xi u_{\xi\xi}^2) + u_\xi^5. \tag{15}$$

It is proved in the article cited that the symmetry algebra for (14) is a direct sum of the symmetry algebras of (15) and of the equation obtained from (15) by replacing ξ by η.

Let us look for boundary conditions of the form

$$a(u, u_x) = 0, \tag{16}$$

for equation (14) compatible with the symmetry

$$\begin{aligned} u_\tau = u_{\xi\xi\xi\xi\xi} &+ 5(u_{\xi\xi}u_{\xi\xi\xi} - u_\xi^2 u_{\xi\xi\xi} - u_\xi u_{\xi\xi}^2) + u_\xi^5 - u_{\eta\eta\eta\eta\eta} - \\ &5(u_{\eta\eta}u_{\eta\eta\eta} - u_\eta^2 u_{\eta\eta\eta} - u_\xi u_{\xi\xi}^2) - u_\eta^5. \end{aligned} \tag{17}$$

Rather simple but tediously long computations lead to the following statement.

Theorem. *Boundary conditions (16) compatible with the symmetry (17) (and then compatible with integrability) are either of the form* $u_x + c\exp(-u)|_{x=0} = 0$ *or* $u_x + c\exp(u/2) \pm \exp(-u)|_{x=0} = 0$, *where* c *is arbitrary.*

Notice that all equations above are invariant under the reflection-type symmetry $x \to -x$. It is unexpected that equations which do not admit any reflection symmetry nevertheless admit, nevertheless, boundary conditions compatible with integrability. For instance, the famous KdV equation

$$u_t = u_{xxx} + 6u_x u \tag{18}$$

is consistent with the boundary condition

$$u = 0|_{x=0},\ u_{xx}|_{x=0} = 0. \tag{19}$$

It implies immediately that the boundary-value problem

$$u_t = u_{xxx} + 6u_x u, \qquad u = 0|_{x=0}$$

with the Dirichlet-type condition at the axis $x = 0$ admits an infinite-dimensional set of "explicit" finite-gap solutions.

This work was partially supported by Russian Foundation of Fundamental Research (Grant 93-011-165) and International Scientific Foundation (Grant RK-2000).

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