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Representations of Algebras Associated with a Möbius Transformation

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Abstract

The Hilbert space representations of a class of commutation relations associated with a Möbius transformation is studied using results on convergence of continued fractions.

1 Introduction

In this paper we study families of Hilbert space operators satisfying commutation relations

$$AB = B(aI + cA)(bI + dA)^{-1}.$$

We say that relations of this form are of a continued fraction type, and the algebras defined by one or several such commutation relations will be called algebras of continued fraction type.

Families of operators satisfying given commutation relations are called representations of those commutation relations. In adition to numerous articles on mathematical physics, we would like to mention the monographs by Jørgensen [5], Putnam [11], Samoilenko [13] and Shmüdgen [14] where verious aspects of representations of commutation relations are considered and many useful references can be found. The classification of representations of commutation relations up to unitary equivalence is a classical problem. One of the most effective techniques used to solve this problem, is to try to transform the given relations to certain other relations, with which one can associate some dynamical system acting on the spectrum of some family of commuting operators. The orbits of the dynamical system would describe the irreducible families of operators, and the operators of an arbitrary representation would then be constructed as the direct integral of the irreducible representations (see Mackey [7] and others). The possibility to describe all irreducible representations just by orbits of the dynamical system depends on the ergodic properties of the system. The dynamical system should not possess an invariant ergodic non-atomic measure. The existence of a measure with such properties is an indication that the commutation relations are not of type I, that is there exist representations of the commutation relations generating a W*-algebra which is a factor of not type I (see Murray, von Neumann [8]). In order to decide the existence or non-existence of such a measure, one has to understand the behaviour of orbits of the dynamical system. In particular, if there exists a measurable section, i.e. a Borel set intersecting each orbit in exactly one point, then there is no such measure. We use the approach outlined above as developed by Vaisleb, Samoilenko [17].

From the methodological side we would like to draw attention to the connection between continued fractions, dynamical systems and algebras with generators satisfying commutation relations. The dynamical systems appearing in this article are generated by a single Möbius transformation of the complex plain, and are therefore related directly to continued fractions.

The results in this paper are also related to results on the representations of C^*-dynamical systems (see e.g. Effros, Hahn [3], Pedersen [10], Tomiyama [16] and references there). We leave the detailed analysis of this relation to a future publication.

2 Continued fractions

We shall need some basic facts on Möbius transformations

$$F(z) = \frac{a + cz}{b + dz}$$

where the coefficients a, b, c, d are fixed complex numbers satisfying $ad - bc \neq 0$ and $d \neq 0$. The function F is a one-to-one mapping of the extended complex plane $\overline{\mathbf{C}}$ onto itself and the inverse mapping $F^{-1}(z) = \frac{-a + bz}{c - dz}$ is also a Möbius transformation with the same restrictions on the coefficients as for F. As a general reference on continued fractions and Möbius transformations we mention [4], Chapter 3.

We are interested in the behaviour of the iterates of F

$$F^{\circ n}(z_0) = F \circ ... \circ F(z_0) \qquad \text{(composition } n \text{ times)}$$

for large positive integers n and different initial points z_0.

The dynamical behaviour of the iterates has to do with the fixed points of F, i.e. the points z such that $F(z) = z$. If $(b - c)^2 + 4ad = 0$ then F has one fixed point $\xi_1 = (c - b)/2d$, and if $(b - c)^2 + 4ad \neq 0$ then F has two fixed points ξ_1 and ξ_2 given by $\frac{c - b \pm \sqrt{(b - c)^2 + 4ad}}{2d}$ (See *Case* 2 bellow for the particular choice of indexes). To get a simple picture of the behaviour of the dynamical system F we make a conformal conjugation of F to a system T by means of a function ϕ which we describe below. This means that $T = \phi \circ F \circ \phi^{-1}$. Then $T^{\circ n} = \phi \circ F^{\circ n} \circ \phi^{-1}$ and $F = \phi^{-1} \circ T \circ \phi$. We distinguish between two cases.

Case 1: F has one fixed point ξ_1.

Then we define ϕ by $\phi(z) = 1/(z - \xi_1)$, i.e. we move the fixed point to infinity. Then T becomes $T(w) = w + l$ where $l = 2d/(b + c)$ and l is different from zero and infinity. From this we see that the nth iterate of T, $T^{\circ n}(w_0)$, tends to infinity with n for every initial point w_0.

The conclusion is that *the iterates $F^{\circ n}(z_0)$ converge to the fixed point ξ_1 for every initial point z_0.*

It may be remarked that in this case $F'(\xi_1) = 1$, i.e. ξ_1 is an indifferent fixed point.

Case 2: F has two fixed points ξ_1 and ξ_2.

Let us choose the indices for ξ_1 and ξ_2 so that $|c - \xi_1 d| \leq |c - \xi_2 d|$. Define ϕ by

$$\phi(z) = \frac{z - \xi_1}{z - \xi_2}.$$

Now the fixed point ξ_1 is moved to zero and ξ_2 to infinity. Then T becomes $T(w) = qw$, where $q = \dfrac{c-\xi_1 d}{c-\xi_2 d} = \dfrac{b+\xi_2 d}{b+\xi_1 d}$. Notice that $|q| \le 1$ and $q \ne 1$. It can be proved that $|F'(\xi_1)| = |q| \le 1$ and $|F'(\xi_2)| = |q|^{-1} \ge 1$. In *Case* 2 we have to distinguish between two cases.

Case 2.1: $|q| < 1$.

Then ξ_1 is an attractive and ξ_2 is a repulsive fixed point. Since $T^{\circ n}(w_0) = q^n w_0$ tends to 0 for every finite initial point w_0, we conclude that *the iterates $F^{\circ n}(z_0)$ tend to ξ_1 for every initial point $z_0 \ne \xi_2$.*

Case 2.2: $|q| = 1$, $q \ne 1$.

If w_0 is different from zero and infinity, then the points $T^{\circ n}(w_0) = q^n w_0$ never converge. Instead they move around the circle with centre at the origin and radius $|w_0|$. In fact, each iteration means multiplication by q, and since $|q| = 1$ this gives a rotation by the angle $\arg(q)$. The interpretation for the dynamical system F is as follows. *If the initial point z_0 is different from ξ_1 and ξ_2 then the iterates $F^{\circ n}(z_0)$ do not converge. Instead they move around the circle*

$$C(z_0): \qquad \left|\frac{z-\xi_1}{z-\xi_2}\right| = \left|\frac{z_0-\xi_1}{z_0-\xi_2}\right|$$

as described above by the dynamics of the conjugate system $T = \phi \circ F \circ \phi^{-1}$. The set $C(z_0)$ is one of the conjugate circles to the fixed points ξ_1 and ξ_2.

Motivated by the investigation above we say that F is of *convergence type* in *Case* 1 and *Case* 2.1 and of *divergence type* in *Case* 2.2. This means that the iterates $F^{\circ n}$ always converge if F is of convergence type and that they move around on circles in the divergence case. It is easy to see that F^{-1} is of convergence type if and only if F is of convergence type.

Example 1 *If $c = 0$ and $d = 1$, then $a \ne 0$. In this case F is of divergence type if and only if $\xi_1 \ne \xi_2$ and $1 = |q| = |\xi_1/\xi_2|$, i.e. $|\xi_1| = |\xi_2|$. When a and b are real, $c = 0$ and $d = 1$, then F is of convergence type if and only if $b \ne 0$ and $b^2 + 4a \ge 0$.*

The following proposition clarifies the situation with periodic points of the Möbius transformation F.

Proposition 1 *The mapping F has periodic points different from the fixed points if and only if F is of divergence type and the number q introduced in Case 2 is an nth root of unity for some positive integer n. Then all points different from the fixed points have the same period.*

We say that a Möbius transformation F is of *periodic divergence type* if it is of divergence type and the number q introduced above is an nth root of unity for some positive integer n.

When F is of divergence type and has no periodic points (except the fixed points), there exists an invariant measure satisfying an ergodic property in the following sense.

Proposition 2([15]) *Assume that F is of divergence type and that the number q in Case* 2.2 *is not an nth root of unity for any positive integer n. Let z_0 be an arbitrary complex number different from the fixed points of F, and $C(z_0)$ the circle introduced in Case* 2.2. *Then there exists a probability measure μ_{z_0} without point masses, with support $C(z_0)$ and such that for all Borel sets E in $\overline{\mathbf{C}}$*

$$\mu_{z_0}(E) = \mu_{z_0}(F^{-1}(E)) \textbf{ (invariance property)}$$

and

$$F(E) = E \textit{ implies that } \mu_{z_0}(E) \textit{ is either } 0 \textit{ or } 1 \textbf{ (ergodic property).}$$

An *orbit* of F is a set $\{F^{\circ n}(z_0) : n \in \mathbf{Z}\}$ where $z_0 \in \overline{\mathbf{C}}$. If n is negative $F^{\circ n}$ denotes n iterations of the inverse function F^{-1}. It is an important question for the understanding of the ergodic properties of F whether there exists a Borel set in $\overline{\mathbf{C}}$ such that any orbit of F meets this set in exactly one point. Such a set is called a *measurable section* of F. The answer is provided by the following theorem.

Theorem 1 *The Möbius transformation F has a measurable section if it is of convergence type. If F is of divergence type it has a measurable section if and only if the number $q = (c - \xi_1 d)/(c - \xi_2 d)$ introduced in Case* 2 *is an nth root of unity for some positive integer n.*

Proof. We first treat the convergence case. It is easy to find a measurable section for the conjugate system T. Let us first check *Case* 2.1. Then $T(w) = qw$ where $|q| < 1$. The fixed points of T are 0 and ∞. A measurable section for T is, for instance, the union of the fixed points of T and the set $\{w : |q| < |w| \leq 1\}$. This means that a measurable section for F is the union of the fixed points for F and the set

$$\left\{ z \in \mathbf{C} : |q| < \left| \frac{z - \xi_1}{z - \xi_2} \right| \leq 1 \right\}.$$

In *Case* 1 a measurable section of the conjugate system $T(w) = w + l$, $l = 2d/(b + c)$, is for instance the union of ∞ (the only fixed point of T for $d \neq 0$) and an infinite strip of width $|l|$ bounded by two lines perpendicular to the vector l. The first line passes through 0 and il, and is included in the strip. The second line is excluded from the strip, and is obtained by adding l to the first line, i.e. it passes through the points l and $(i + 1)l$.

Let us now treat the divergence case. If q is an nth root of unity, and n is the smallest of such integers, then all points except the fixed points have period n, and a measurable section for $T(w) = qw$ is, for instance, the union of $\{0, \infty\}$ and the angular sector $\left\{ w \in \mathbf{C} : 0 \leq \arg(w) < \frac{2\pi}{n} \right\}$. The corresponding measurable section for F is the union of the fixed points of F and the set $\left\{ z \in \mathbf{C} : 0 \leq \arg\left(\frac{z - \xi_1}{z - \xi_2}\right) < \frac{2\pi}{n} \right\}$. When q is not a root of unity, Proposition 2 implies that there does not exist any measurable section. Indeed, if such a measurable section would exist, then any measure, satisfying both the invariance property and the ergodic property, would have to be concentrated on one orbit of the dynamical system which contradicts Proposition 2. □

3 Classification of representations

3.1. In this section we consider the problem of classification up to unitary equivalence of pairs (A, B) of Hilbert space operators consisting of a self-adjoint operator A and a unitary operator B satisfying the relation

$$AB = aB(bI + A)^{-1} \tag{1}$$

where the parameters a and b are real numbers and $a \neq 0$. The point spectrum $\sigma_p(A)$ of A is assumed not to contain $-b$. This in particular implies that $E_A(\{-b\}) = 0$, where $E_A(\cdot)$ denotes the resolution of the identity of A. Henceforth, $\mathcal{H}_\lambda$ denotes the eigensubspace of the operator A associated with an eigenvalue λ. We will also use the notation $F_{a,b,c,d}(z) = \dfrac{a+cz}{b+dz}$ and $F_{a,b}(z) = F_{a,b,0,1}(z) = \dfrac{a}{b+z}$.

Since we do not assume the operator A to be bounded we must specify what is meant by relation (1) for an unbounded A.

We will accept the following definition (see [17]).

Definition 1 *The self-adjoint operator A and the unitary operator B in the Hilbert space $\mathcal{H}$ are said to be a representation of the relation* (1) *if*

$$E_A(\delta)B = BE_A(F_{a,b}^{-1}(\delta)) \tag{2}$$

for any Borel set δ of $\mathbf{R}^1$.

The following theorem shows that if A is bounded and $-b \notin \sigma(A)$, then this definition and the ordinary one are equivalent.

Theorem 2 ([17]) *For a bounded self-adjoint operator A with $-b \notin \sigma(A)$ and a unitary operator B the following three properties are equivalent:*

(i) *The operators A and B satisfy relation* (1) *in the sense of Definition* 1.

(ii) $AB = aB(bI + A)^{-1}$ *pointwise on $\mathcal{H}$.*

(iii) *For any bounded Borel function $\phi(\cdot) : \mathbf{R}^1 \mapsto \mathbf{R}^1$*

$$\phi(A)B = B\phi(a \cdot (bI + A)^{-1}).$$

We now turn our attention to the spectrum $\sigma(A)$ of A. The next lemma connects properties of the spectrum of A with properties of the mapping $F_{a,b}$.

Lemma 1 (i) *If $\lambda \in \sigma(A) \setminus \{-b\}$, then $F_{a,b}(\lambda) \in \sigma(A) \setminus \{0\}$.*
(ii) *If $\mu \in \sigma(A) \setminus \{0\}$, then $F_{a,b}^{-1}(\mu) \in \sigma(A) \setminus \{-b\}$.*

Proof. With each self-adjoint operator A in a Hilbert space one can associate its resolution of the identity $E_A(\cdot)$. It is well known that a point μ belongs to the spectrum of A if and only if $E_A(\delta) \neq 0$ for any interval δ containing μ [12]. Let $\lambda \in \sigma(A) \setminus \{-b\}$. To prove (i) it is enough to show that $E_A(\delta) \neq 0$ for any interval δ such that $F_{a,b}(\lambda) \in \delta$ and $0 \notin \delta$. Since $F_{a,b}^{-1}(\delta)$ is an interval containing λ and since $\lambda \in \sigma(A) \setminus \{-b\}$, we get $E_A(F_{a,b}^{-1}(\delta)) \neq 0$. This inequality and the relation (2) imply $E_A(\delta) = BE_A(F_{a,b}^{-1}(\delta))B^* \neq 0$.

Now let $\mu \in \sigma(A) \setminus \{0\}$. To prove (ii) it is enough to show that $E_A(\theta) \neq 0$ for any interval θ such that $F_{a,b}^{-1}(\mu) \in \theta$ and $-b \notin \theta$. Since $F_{a,b}(\theta)$ is an interval containing μ and since $\mu \in \sigma(A) \setminus \{0\}$, we get $E_A(F_{a,b}(\theta)) \neq 0$. This inequality and the relation (2) imply $E_A(\theta) = B^* E_A(F_{a,b}(\theta))B \neq 0$. □

For an initial point λ_0, the sets $Orb_{a,b}(\lambda_0) = \{\lambda_n : F_{a,b}(\lambda_n) = \lambda_{n+1}, n = \cdots, -1, 0, 1, \cdots\}$, $Orb^+_{a,b}(\lambda_0) = \{\lambda_n : F_{a,b}(\lambda_n) = \lambda_{n+1}, n = 0, 1, \cdots\}$ and $Orb^-_{a,b}(\lambda_0) = \{\lambda_n : F_{a,b}(\lambda_n) = \lambda_{n+1}, n = \cdots, -2, -1\}$ are called respectively an orbit, a forward orbit and a backward orbit of $F_{a,b}$ passing through λ_0. Clearly $Orb_{a,b}(\lambda) = Orb^+_{a,b}(\lambda) \bigcup Orb^-_{a,b}(\lambda)$.

Remark 1 *From the relation* (2) *it follows that none of the points from* $Orb^+_{a,b}(0) \bigcup Orb^-_{a,b}(-b)$ *can belong to the point spectrum of the operator* A *in a representation of relation* (1). *The situation changes if, for example, we allow* B *to be an isometry.*

Let us define the irreducible representations of the relation (1).

Definition 2 *The representation* (A, B) *of the relation* (1) *is called irreducible if any bounded linear operator commuting with the operators* A *and* B *is a multiple of the identity.*

Let $F_{a,b}$ be of convergence or periodic divergence type. As was described in Section 2 this takes place if and only if $b^2 + 4a \geq 0$ or $q = \dfrac{-b + \sqrt{b^2 + 4a}}{-b - \sqrt{b^2 + 4a}} = \exp(i2\pi\alpha)$ for some rational α. Theorem 1 shows that, in these cases, $F_{a,b}$ does possess a measurable section M.

The following proposition is the key to the description of the irreducible representations both in the convergence and the periodic divergence case.

Proposition 3([15]) *Let* $F_{a,b}$ *be of convergence type or of periodic divergence type. Then, in an irreducible representation* (A, B) *of the relation* (1), *the spectrum of the operator* A *is simple and based on some orbit of the Möbius transformation* $F_{a,b}$, *that is* $E_A(Orb_{F_{a,b}}(\lambda)) = I$ *for some* λ. *If* e_λ *is an eigenvector of the operator* A *corresponding to an eigenvalue* λ, *then* Be_λ *is an eigenvector of the operator* A *corresponding to an eigenvalue* $a/(b + \lambda)$.

We are ready now to give a unitary classification of the irreducible representations of the relation (1).

3.2. We will start with a description of the representations of (1) in the convergence case. In what follows we put $M_{\mathbf{R}} = M \cap \mathbf{R}^1$. If M is a measurable section for F on the complex plane, then $M_{\mathbf{R}}$ is a measurable section for F on the real line.

Theorem 3 *Assume that* $a, b \in \mathbf{R}^1$ *are such that* $F_{a,b}(x) = \frac{a}{b+x}$, $x \in \mathbf{R}^1$, *is a Möbius transformation of convergence type. Let* M *be a measurable section for* $F_{a,b}$ (*see Th.*1) *and* ξ_1 *and* ξ_2 *be the fixed points of* $F_{a,b}$. *Then all irreducible representations of the relation* (1) *are either one-dimensional* $(\dim(\mathcal{H}) = 1)$ *or bounded infinite-dimensional* $(\dim(\mathcal{H}) = \infty)$ *and classified as follows.*

(i) $(\dim(\mathcal{H}) = 1)$. *Each point* $(\xi_k, \phi) \in \{\xi_1, \xi_2\} \times [0, 2\pi), k = 1, 2$, *parametrizes the unique irreducible representation. The operator* A *is multiplication by* ξ_k *and* B *is multiplication by* $\exp(i\phi)$ *in the space* $\mathcal{H} \cong \mathbf{C}$. *The representations corresponding to different points in* $\{\xi_1, \xi_2\} \times [0, 2\pi)$ *are unitarily inequivalent.*

(ii) $(\dim(\mathcal{H}) = \infty)$. *Each* $\lambda \in M_{\mathbf{R}} \setminus \{\xi_1, \xi_2\}$ *parametrizes the unique irreducible representation. It is infinite-dimensional, bounded and defined by the formulas*

$$Ae_j = F^{\circ j}_{a,b}(\lambda)e_j, \quad Be_j = e_{j+1}, \quad j \in \mathbf{Z}$$

where $\{e_j\}_{j\in\mathbf{Z}}$ *is an orthonormal basis consisting of eigenvectors of the operator* A *and* $F^{\circ j}_{a,b}(\lambda) = \frac{a}{b+}\frac{a}{b+}\cdots+\frac{a}{(b+\lambda)}$ *is the value of the approximant of order* j *at the point* λ *for the continued fraction* $\frac{a}{b+}\frac{a}{b+}\cdots+\frac{a}{b+}\cdots$. *The representations corresponding to different* λ *are unitarily inequivalent.*

Any irreducible representation of the relation (1) *is unitarily equivalent to one of the representations defined in* (i) *and* (ii).

Proof. First of all, by Proposition 1 the mapping $F_{a,b}$ has no periodic points except the fixed points ξ_1 and ξ_2. Next, from Proposition 3 we have either $\sigma(A) = \{\xi_1\}$ or $\sigma(A) = \{\xi_2\}$ or $\sigma(A) = \overline{Orb_{F_{a,b}}(\lambda)}$ for some $\lambda \in M_{\mathbf{R}} \setminus \{\xi_1, \xi_2\}$. The last possibility means that the spectrum $\sigma(A)$ is a union of the orbit $Orb_{F_{a,b}}(\lambda)$ and the limit points of $Orb_{F_{a,b}}(\lambda)$ which in the convergence case are exactly the fixed points of $F_{a,b}$. The eigensubspaces $\mathcal{H}_\lambda$ of A are one-dimensional (see Prop. 3). If $\sigma(A) = \{\xi_1\}$ or $\sigma(A) = \{\xi_2\}$, then we get (i). It remains to show that if $\sigma(A) = Orb_{F_{a,b}}(\lambda)$ for some $\lambda \in M_{\mathbf{R}} \setminus \{\xi_1, \xi_2\}$, then the representation is unitarily equivalent to that in (ii). To see this take the unit vector $e_\lambda \in \mathcal{H}_\lambda$ and form the orthonormal basis $e_j = B^j e_\lambda$, $j \in \mathbf{Z}$, of $\mathcal{H}$ consisting of eigenvectors of the operator A. Then in this basis the operators A and B are defined as in (ii). □

The next theorem is a kind of spectral theorem in which an arbitrary representation of the relation (1) is described as being unitarily equivalent to a direct sum or integral of the irreducible representations. The proof can be adapted from the proof of the similar theorem in [17] (see [15]).

Theorem 4 *Assume that* $a, b \in \mathbf{R}^1$ *are such that* $F_{a,b}(x) = \frac{a}{b+x}$, $x \in \mathbf{R}^1$, *is a Möbius transformation of convergence type. Let* M *be a measurable section for* $F_{a,b}$ (*see Th.*1) *and* ξ_1 *and* ξ_2 *be the fixed points of* $F_{a,b}$.

Let (A, B) *be a representation of the relation* (1) *in a Hilbert space* $\mathcal{H}$ (*see Def.* 1). *Then* $\mathcal{H} = \mathcal{H}^{(1)} \bigoplus \mathcal{H}^{(\infty)}$, *where* $\mathcal{H}^{(1)}$ *and* $\mathcal{H}^{(\infty)}$ *are orthogonal subspaces of* $\mathcal{H}$, *invariant with respect to* A *and* B. *The subspace* $\mathcal{H}^{(\infty)}$ *may be represented as* $\mathcal{H}^{(\infty)} = l^2(\mathbf{Z}) \bigotimes H_{(\infty)}$, *where* $H_{(\infty)}$ *is some subspace of* $\mathcal{H}$. *The pair* (A, B) *of operators is unitarily equivalent to*

$$A = \int\limits_{\{\xi_1,\xi_2\}\times[0,2\pi)} \lambda\, dE_1(\lambda,\phi) + \int\limits_{M_{\mathbf{R}}\setminus\{\xi_1,\xi_2\}} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \frac{a}{\lambda} - b & 0 & \ddots & \ddots \\ \ddots & 0 & \lambda & 0 & \ddots \\ \ddots & \ddots & 0 & \frac{a}{b+\lambda} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \bigotimes dE_\infty(\lambda),$$

$$B = \int\limits_{\{\xi_1,\xi_2\}\times[0,2\pi)} \exp(i\phi)\, dE_1(\lambda,\phi) + \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & 0 & 0 & \ddots \\ \ddots & 1 & 0 & 0 & \ddots \\ \ddots & 0 & 1 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \bigotimes I_{H_{(\infty)}}$$

where $E_1(\cdot,\cdot)$ *is some resolution of the identity* (*probability projection valued measure*) *defined on Borel subsets of* $\{\xi_1, \xi_2\} \times [0, 2\pi)$, *taking values in projections onto subspaces of*

$\mathcal{H}^{(1)}$, and $E_\infty(\cdot)$ is some resolution of the identity defined on Borel subsets of $M_{\mathbf{R}}\setminus\{\xi_1,\xi_2\}$, taking values in projections onto subspaces of $H_{(\infty)}$.

It is possible to reformulate the above theorem in multiplication operator form. This also would give a complete set of unitary invariants in terms of the spectral classes of measures and function of multiplicity.

3.3. So far we have delt with the convergence case. Let us describe what happens if $F_{a,b}(x) = \frac{a}{b+x}$ is of periodic divergence type (see Sec. 2).

In the periodic divergence case, the Möbius transformation again posseses a measurable section (see Th.1). So both the theorem describing irreducible pairs and the structure theorem may be proved by arguments similar to those used in Theorems 3 and 4.

Hence, we will just formulate the theorems and omit the proofs.

In the periodic divergence case all irreducible representations of (1) are finite-dimensional and described, up to unitary equivalence, in the following theorem.

Theorem 5 *Assume that $a, b \in \mathbf{R}^1$ are such that $F_{a,b}(x) = \frac{a}{b+x}$, $x \in \mathbf{R}^1$, is a Möbius transformation of periodic divergence type. Let $M_{\mathbf{R}}$ be a measurable section for $F_{a,b}$ on the real line, ξ_1 and ξ_2 be the fixed points of $F_{a,b}$, the number ξ_1/ξ_2 be an nth root of unity for some integer $n > 1$ and n be the smallest of such integers.*

(i) *If $n \geq 3$, then all irreducible representations of the relation* (1) *are n-dimensional ($\dim(\mathcal{H}) = n$). Each $(\lambda, \phi) \in M_{\mathbf{R}} \times [0, 2\pi)$ parametrizes, up to unitary equivalence, a unique irreducible representation, defined by the formulas*

$$
\begin{aligned}
Ae_j &= (\underbrace{\frac{a}{b} + \frac{a}{b} + \cdots + \frac{a}{b+\lambda}}_{\text{approximant of order j}})e_j, \qquad 0 \leq j \leq n-1 \\
Be_j &= \begin{cases} e_{j+1} & \text{if } 0 \leq j \leq n-2 \\ \exp(i\phi)e_0 & \text{if } j = n-1 \end{cases}
\end{aligned}
$$

where $\{e_0, \ldots, e_{n-1}\}$ is an orthonormal basis consisting of eigenvectors of the operator A.

(ii) *If $n = 2$, then $\xi_2 = -\xi_1$, $b = 0$, $F_{a,b}(x) = a/x$. If $a > 0$ then $\{\xi_1,\ \xi_2\} \subset \mathbf{R}^1$, and if $a < 0$ then $\{\xi_1,\ \xi_2\} \subset i\mathbf{R}^1$.*

If $\{\xi_1,\ \xi_2\} \subset i\mathbf{R}^1$, then all irreducible representations are two-dimensional and described in (i) *with $n = 2$.*

If $\{\xi_1,\ \xi_2\} \subset \mathbf{R}^1$, then all irreducible representations are either two-dimensional or one-dimensional. All two-dimensional representations are as described in (i) *with $n = 2$ except that now they are parametrized by the points $(\lambda, \phi) \in (M_{\mathbf{R}} \setminus \{\xi_1, \xi_2\}) \times [0, 2\pi)$. The one-dimensional representations are parametrized, up to unitary equivalence, by the points $(\xi_k, \phi) \in \{\xi_1, \xi_2\} \times [0, 2\pi)$, $k = 1, 2$. The operator A is multiplication by ξ_k and B is multiplication by* $\exp(i\phi)$ *in the space $\mathcal{H} \cong \mathbf{C}$.*

The spectral theorem in the periodic divergence case can be formulated as follows.

Theorem 6 *Assume that $a, b \in \mathbf{R}^1$ are such that $F_{a,b}(x) = \frac{a}{b+x}$, $x \in \mathbf{R}^1$, is a Möbius transformation of periodic divergence type. Let $M_{\mathbf{R}}$ be a measurable section for $F_{a,b}$ on* *the real line, ξ_1 and ξ_2 be the fixed points of $F_{a,b}$, the number ξ_1/ξ_2 be an nth root of unity for some integer $n > 1$ and n be the smallest of such integers.*

Let (A, B) be a representation of the relation (1) in a Hilbert space $\mathcal{H}$

(i) *If $n \geq 3$ or $n = 2$ and $\{\xi_1,\ \xi_2\} \subset i\mathbf{R}^1$, then there is a subspace $H_{(n)}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathbf{C}^n \bigotimes H_{(n)}$, and the pair of operators (A, B) is unitarily equivalent to*

$$A = \int\limits_{M_{\mathbf{R}}\setminus\{\xi_1,\xi_2\}\times[0,2\pi)} \begin{pmatrix} \lambda & 0 & \ddots & 0 \\ 0 & \frac{a}{b+\lambda} & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & \underbrace{\cfrac{a}{b+\cfrac{a}{b+\ddots+\cfrac{a}{b+\lambda}}}}_{n-1} \end{pmatrix} \bigotimes dE_n(\lambda, \phi)$$

$$B = \int\limits_{M_{\mathbf{R}}\setminus\{\xi_1,\xi_2\}\times[0,2\pi)} \underbrace{\begin{pmatrix} 0 & 0 & \ddots & \exp(i\phi) \\ 1 & 0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 0 \end{pmatrix}}_{\text{n}\times\text{n matrix}} \bigotimes dE_n(\lambda, \phi)$$

where $E_n(\cdot, \cdot)$ is some resolution of the identity defined on Borel subsets of $(M_{\mathbf{R}}\setminus\{\xi_1, \xi_2\}) \times [0, 2\pi)$, taking values in projections onto subspaces of $H_{(n)}$.

(ii) *If $n = 2$ and $\{\xi_1,\ \xi_2\} \subset \mathbf{R}^1$, then there exist orthogonal subspaces $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$ and $H_{(2)}$ such that $\mathcal{H} = \mathcal{H}^{(1)} \bigoplus \mathcal{H}^{(2)}$, $\mathcal{H}^{(2)} = \mathbf{C}^2 \bigotimes H_{(2)}$ and the pair (A, B) is unitarily equivalent to*

$$A = \int\limits_{\{\xi_1,\xi_2\}\times[0,2\pi)} \lambda\, dE_1(\lambda, \phi) + \int\limits_{M_{\mathbf{R}}\setminus\{\xi_1,\xi_2\}\times[0,2\pi)} \begin{pmatrix} \lambda & 0 \\ 0 & \frac{a}{\lambda} \end{pmatrix} \bigotimes dE_2(\lambda, \phi)$$

$$B = \int\limits_{\{\xi_1,\xi_2\}\times[0,2\pi)} \exp(i\phi)\, dE_1(\lambda, \phi) + \int\limits_{M_{\mathbf{R}}\setminus\{\xi_1,\xi_2\}\times[0,2\pi)} \begin{pmatrix} 0 & \exp(i\phi) \\ 1 & 0 \end{pmatrix} \bigotimes dE_2(\lambda, \phi)$$

where $E_1(\cdot, \cdot)$ is some resolution of the identity defined on Borel subsets of $\{\xi_1, \xi_2\} \times [0, 2\pi)$, taking values in projections onto subspaces of $\mathcal{H}^{(1)}$, and $E_2(\cdot, \cdot)$ is some resolution of the identity defined on Borel subsets of $(M_{\mathbf{R}} \setminus \{\xi_1, \xi_2\}) \times [0, 2\pi)$, taking values in projections onto subspaces of $H_{(2)}$.

4 Some applications, problems and remarks

4.1. Using the convergence properties of the Möbius transformations on $\mathbf{C}$ described in Section 2, all results from Section 3 can be extended to describe the representations (A, B)

of the commutation relation (1) in which A is a normal operator, B is unitary, and the parameters a and b are in general complex numbers. We refer to [15] for details.

4.2. The problem of classification of representations of the more general commutation relations

$$AB = B(aI + cA)(bI + dA)^{-1} \tag{3}$$

or of the relations

$$\begin{aligned} A_1A_2 &= A_2A_1 \\ A_1B &= B\, Re((aI + c(A_1 + iA_2))(bI + d(A_1 + iA_2))^{-1} \\ A_2B &= B\, Im((aI + c(A_1 + iA_2))(bI + d(A_1 + iA_2))^{-1} \end{aligned} \tag{4}$$

can be solved analogously to the case when $c = 0, d = 1$. All proofs and results are essentially the same except that the formulas become dependent on four parameters a, b, c, d. In particular, the commutation relations are of type I exactly when either $\left|\frac{c-\xi_1 d}{c-\xi_2 d}\right| \neq 1$ or $\frac{c-\xi_1 d}{c-\xi_2 d}$ is a root of unity. Here ξ_1, ξ_2 are the fixed points of the Möbius transformation $F_{a,b,c,d}(z) = \frac{a+cz}{b+dz}$. To get the formulas for representations in this case, one should simply replace everywhere $\frac{a}{b+z}$ by $\frac{a+cz}{b+dz}$.

4.3. Theorems 3, 4, 5 and 6 were obtained under the assumption that the corresponding continued fraction is of convergence or periodic divergence type. In particular, the conclusion can be drawn that in this case the commutation relations of continued fraction type are of type I. The situation changes dramatically in the non-periodic divergence case. The main difficulty is that in the non-periodic divergence case, as described in Section 2, there exists a non-atomic measure which is invariant and ergodic with respect to the corresponding Möbius transformation. Such a measure can not be based on a single orbit. The existence of such a measure makes possible the construction of a factor-representation of the relation (3) which is not of type I (see [8]). This means that the corresponding commutation relations are not of type I.

4.4. The problem of classification of the representations, by selfadjoint operators, of polynomial commutation relations of the third degree, linear with respect to one of the variables, was considered in [9], [2].

Direct application of our results gives a solution to the problem of classification of pairs (A, B) consisting of a self-adjoint operator A and a unitary operator B satisfying a third degree commutation relation of the form $a_1ABA + a_2AB + a_3BA + a_4B = 0$. To see this simply multiply (3) by $bI + dA$ from the right and then rename the coefficients.

4.5. The algebra of polynomials on the closed quantum unit disc is one of the many examples appearing in both mathematical and physical literature. It can be defined as a $*$-algebra generated by two elements z and z^* satisfying the following commutation relation:

$$zz^* - z^*z = \mu(I - zz^*)(I - z^*z) \qquad (0 < \mu < 1). \tag{5}$$

In Section 3 of [6] the irreducible representations of (5) have been classified up to unitary equivalence.

Let us consider (5) in the light of the commutation relations of continued fraction type. Let $z = CU$ be the left polar decomposition of z, where C is positive operator, and U is a partial isometry. Then the relation (5) can be rewritten as

$$CU = U((1+\mu)C - \mu I)(\mu C + (1-\mu)I)^{-1}. \tag{6}$$

Note that this is exactly a commutation relation of continued fraction type corresponding to the Möbius transformation $F_{a,b,c,d}(z) = \dfrac{-\mu + (1+\mu)z}{(1-\mu)+\mu z}$ with parameters $a = -\mu$, $b = 1-\mu$, $c = 1+\mu$, $d = \mu$. It is easy to check that this Möbius transformation has only one fixed point $\xi = \xi_1 = \xi_2 = 1$ for any $\mu \neq 0$. Hence, it is of convergence type as described in *Case* 1 in Section 2. Therefore, the commutation relation (5) is of Type I for any $\mu \neq 0$ (see Subsec. 4.3).

Theorem 3 and Subsection 4.2 provide us with unitary classification of all irreducible representations of (6), with not-necessarily positive self-adjoint C, but with unitary U. In particular, all of them are either one-dimensional or infinite-dimensional and the resolution of the identity of C is based on an orbit of $F_{a,b,c,d}$. All representations are described as direct integrals of the irreducible representations as in Theorem 4.

For $0 < \mu < 1$ any infinite orbit of $F_{a,b,c,d}$ consists of both positive and negative numbers. This implies that C can be positive in the irreducible representation of (6) if and only if this representation is one-dimensional ($\mathcal{H} = \mathbf{C}$). Therefore, all irreducible representations of (6) with positive C are one-dimensional and unitarily equivalent to one of the representations of the form $Cf = f,\ Uf = \exp(i\phi)f,\quad f \in \mathbf{C},\ \phi \in [0, 2\pi)$.

The operator U in the polar decomposition of z does not have to be unitary. In general, it may be just a partial isometry. However, one can prove, using the relation (5), that in the irreducible representation of (6), the operator U must be an isometry. This is exactly the case for the infinite-dimensional irreducible representation of (5) described in [6].

4.6. Finally, we would like to mention that it is an interesting problem to describe, up to unitary equivalence, families of operators $\{A_j\}$, B, consisting of finitely or infinitely many mutually commuting self-adjoint operators A_j and one unitary operator (partial isometry) B, satisfying several commutation relations of continued fraction type

$$A_j B = a_j B (b_j I + A_j)^{-1} \qquad \text{for all } j.$$

In particular, one would have to answer the question: for which parameters a_i and b_i are these commutation relations of type I?

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