

On Representations of $*$ -Algebras in Mathematical Physics

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1. The theory of representations of \mathcal{A} ($*$ -homomorphisms $\pi: \mathcal{A} \mapsto L(H)$ of a $*$ -algebra \mathcal{A} into a $*$ -algebra $L(H)$ of all bounded operators or into a $*$ -algebra of unbounded operators in a separable complex Hilbert space H) is important both in mathematics and in physical applications.

The $*$ -representations of \mathcal{A} (a local object) contain information about the structure of the algebra itself as well as about the structure of its dual object. Studying a particular class of representations of \mathcal{A} in general by unbounded operators allows to define the corresponding non-commutative manifold: a C^* -algebra \mathcal{A} (a global object) which possesses this class of representations.

A choice of a particular representation $\pi(\cdot)$ corresponds to the choice of a model with observables $A_k = \pi(a_k)$ ($k = 1, \dots, n$) obeying the relations

$$P_k(A_1, \dots, A_n) = 0 \quad (k = 1, \dots, m). \quad (1)$$

A family of self-adjoint (bounded or unbounded) operators $(A_k)_{k=1}^n$ satisfying (1) completely define the representation $\pi(\cdot)$. We will call such families by representations of relations (1).

This talk is a survey of some results by the authors and their collaborators on the structure of pairs and families of bounded and unbounded operators satisfying commutation relations which can be studied with the help of corresponding one-dimensional or multidimensional discrete dynamical systems.

The method of solving operator problems by using d.s. ascends to the classical papers [10, 11] (see, e.g., [6] and the bibliography therein) and is of wide use now (see, for example, [21, 5, 20]). The new aspects in the talk are related to problems concerning:

transition from representations by unbounded operators to representations by bounded operators;

need to consider isometries and centered partial isometries in the operator part of the problem.

use of topological properties of dynamical systems, which are in general not one-to-one, to solve operator problems;

2. Let us consider the operator equation

$$AB = BF(A), \quad (2)$$

where $A = A^*$ is a self-adjoint (unbounded) operator, B is a bounded or an unbounded closed operator and $F(\cdot)$ is a continuous real valued function on the real line \mathbb{R} .

To make sense out of relation (2), consider the polar decomposition of the operator $B = U|B|$ and the projection $P = \text{sign } |B|$ onto the initial space of the partial isometry U .

Definition 1 (see [1, 15]). *We say that operators A and B satisfy relation (2) if the following relations for the operators C and X hold*

$$E_C(\Delta)UP = UE_C(F^{-1}(\Delta))P, \quad [E_{|B|}(\Delta), E_C(\Delta)] = 0, \quad (3)$$

$$\Delta, \Delta' \in \mathcal{B}(\mathbb{R}^1).$$

3. Given a function $F(\cdot)$ such that there exists a stationary point $x_0 = F(x_0)$, the description, up to unitary equivalence, of all solutions of equation (2) is a very difficult (“wild”) problem if no *-conditions are imposed on the operator B . It contains as a subproblem the problem of description up to unitary equivalence of pairs $A = x_0I, B$, i.e., the standard wild *-problem of the description of all pairs of self-adjoint operators (all non-self-adjoint B) up to unitary equivalence, which itself contains a subproblem of the description, up to unitary equivalence, of any finite families of self-adjoint operators (see [9]).

Thus, to describe the structure of all (irreducible) solutions of relation (2), some additional conditions must be required. We will assume some algebraic relations between B and B^* (involution condition). But if one requires, for example, that the operator B satisfies the relation $[BB^*, B^*B] = 0$ (i.e., that B is weakly centered, see [2, 18]), the description problem for such operators is “wild”, i.e., contains in itself the standard wild problem [8]. The class of centered operators, i.e., operators for which $B^j(B^*)^j, (B^*)^k B^k, k = 1, 2, \dots$ form a commuting family [12] is not already “wild”. A description of such operators is yet complicated, since there exist centered operators generating factors not of type I ; however, these factors are approximately finite.

In our talk, situations occur, when the relation connecting B and B^* is of the type I (for such operators there exists the corresponding structure theorem) and when the relation is not of the type I , but in all cases B is supposed to be centered.

4. In the first part of the talk, following [14, 15], we give a solution to the above discussed problem for a self-adjoint B , for a unitary B (see also [23, 22]) and for a normal B . Then we treat the case when B is an isometry, coisometry, or centered partial isometry. The case of arbitrary centered operator can be reduced to the case of centered partial isometry (which may occur unitary, isometry or co-isometry) by writing the polar decomposition $B = |B|U$, where U is centered partial isometry, and $|B|$ commutes with A . In particular, the investigation of a relation of the form $XX^* = f(X^*X)$ can be reduced to the case of centered partial isometry, since X is centered here (on representations of the relation $XX^* = f(X^*X)$ see [22, 4] etc.).

In the second part of the talk we generalize the obtained results to the case of multi-dimensional dynamical system and families of operators (see [23, 17]).

In the third part we consider a number of examples illustrating the use of the results from the previous exposition.

A study of these problems is necessary in an investigation of various examples of *-algebras appearing in the literature on Mathematical Physics (see [3, 7, 19, 13, 24, 16] etc.).

In the talk we study their representations with help of a dynamical systems technique. Some results on their representations are obtained by the authors, another ones are known (see references below) but can be obtained using the d.s. formalism.

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