An efficient iteratively reweighted L1-minimization for image reconstruction from compressed sensing

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Abstract. We proposed a simple and efficient iteratively reweighted algorithm to improve the recovery performance for image reconstruction from compressive sensing (CS). The numerical experiential results demonstrate that the new proposed method outperforms in image quality and computation complexity, compared with standard l_1-minimization and other iteratively reweighted l_1-algorithms when applying for image reconstruction from CS.

Keywords: Image reconstruction, Compressive sensing, l_1-Minimization, Reweighted algorithm.

1 Introduction

Compressive sensing theory presents [1-2] that a sparse signal can be reconstructed from a small number of random linear measurements using l_1 optimization (instead of l_0 optimization) algorithm, under some condition (such as mutual coherence (MC)[3-4], restricted isometry property/Condition (RIP or RIC) [5], or null space property (NSP) [3, 6]). Recent studies indicate that the iteratively reweighted l_1-minimization does have an advantage over standard l_1-minimization in many situations [7-10] to find the sparest solution of an underdetermined linear system, which can be formulated as weighted l_1-problems (WP_1) as follows.

$$(WP_1) \quad \mathbf{x}^{(i)} = \min_{\mathbf{x}\in R^{N\times 1}} \left\|\mathbf{W}^{(i)}\mathbf{x}\right\|_{l_1} \quad subject \quad to \quad \mathbf{Ax} = \mathbf{y} \quad (1)$$

here $\mathbf{w}^{(i)} = \mathrm{diag}(w^{(i)})$ and $w^{(i)} = (w_1^{(i)}, w_2^{(i)}, \cdots, w_N^{(i)})^T \in \mathrm{R}^{N+}$ (means positive real number) are the vector of weights determined by the previous iterate $\mathbf{x}^{(i-1)} = (x_1^{(i-1)}, \cdots, x_j^{(i-1)}, \cdots, x_N^{(i-1)})^T \in \mathrm{R}^N$.

Mathematically speaking, the weight is used to drive $\mathbf{x}^{(i)}$ to it's the sparsest solution (the solution of l_0-minimization) via penalizing the components of $\mathbf{x}^{(i)}$ using minimizing the weighted l_1-norm. In other words, the target of (WP_1) is to select a solution which is approximate to the solution of l_0-minimization from its all possible solutions. To this end, we need to specify a merit function for sparsity. Using such a

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function may drive the variable $\mathbf{x}$ to become sparse provided that a sparse solution exists. Clearly, there exist a vast number of merit functions for sparsity [11].

Recently, CS-based image/video sampling and compression has been studied in [17-18]. These methods aim to reduce the number of CS measurements and thus improve the coding efficiency. In this paper work, we proposed a new and fast iteratively reweighted l_1 -minimization algorithm for finding the sparest solution of an underdetermined linear system and extended our work to two dimensional signal (image) and measure the reconstruction quality and computation complexity, in comparison to classical l_1 -minimization and other iteratively reweighted l_1 -minimization algorithms.

2. UNIFIED STRUCTURE OF REWEIGHTED l_1 -MINIMIZATION

Using an iterative algorithm to construct the weights $\mathbf{w}^{(i)}$ tends to allow for successively better estimation of the nonzero coefficient locations. The central idea of (WP_1) is to define a weight $\mathbf{w}^{(i)}$ based on the previous iterate $\mathbf{x}^{(i-1)}$, solve (WP_1) with the weight, and then use its solution to define a new weight $\mathbf{w}^{(i+1)}$. The structure of iteratively reweighted l_1-minimization is as follows.

1) Set the iteration count i to zero and $w_j^{(0)} = 1, j = 1, 2, \cdots, N$

2) Solve (WP_1) (formula (1))

3) Terminate on convergence or when i attains a specified maximum number of iterations $i_{\max}$. Otherwise, increment i.

4) Update the weights with the equation $\mathbf{W}^{(i)} = f(\mathbf{x}^{(i-1)})$ and the then go to step 2.

The weight of a reweighted l_1-minimization is yielded by merit function as follows

$$w_j^{(i+1)} = f(x_j^{(i)}) = \nabla g_j(\left|x_j^{(i)}\right| + \varepsilon^{(i)}) \qquad (2)$$

or by support set (T_S)

$$w_j^{(i+1)} = \begin{cases} C_1, & \left|x_j^{(i)}\right| \in T_S \\ C_0, & others \end{cases} \qquad (3)$$

here $g_j(\left|x_j^{(i)}\right| + \varepsilon^{(i)})$ is a merit function, ∇ is a gradient operator, $\varepsilon^{(i)}$, C_1 and C_0 are constant. This leads weighted l_1-minimization (1) to the approximation problem of (P_0).

For example, the function $\mathrm{G}_\varepsilon(\mathbf{x}) = \sum \log(\left|x_j\right| + \varepsilon)$ was used by Gorodnitsky and Rao [19] to design the FOCUSS algorithm, and E. J. Candès [7] to design reweighted l_1-minimization, for sparse signal reconstruction. Considering the limited space of this paper, we summarize the existing algorithms as follows, which are based on a merit function or support set for sparsity.

1)E. J. Candès [7](WL1FIX)

$$g_j(|x_j|+\varepsilon)=\log(|x_j|+\varepsilon); \quad w_j^{(i+1)}=\frac{1}{|x_j^{(i)}|+\varepsilon^{(i)}} \quad (4)$$

2) D. Wipf [8](WR2REG)

$$g_j(|x_j|+\varepsilon)=\log(x_j^2+\varepsilon); \quad w_j^{(i+1)}=\frac{1}{(x_j^{(i)})^2+\varepsilon^{(i)}} \quad (5)$$

3) Y. Wang [11](ISD)

$w_j^{(i+1)}=\begin{cases}1, & |x_j^{(i)}|\in T_S \\ 0, & others\end{cases}$, $g_j(|x_j|+\varepsilon)$ is a Support Set (T_S)

4) L. Qin [9](RISD)

$w_j^{(i+1)}=\begin{cases}\frac{1}{|x_j^{(i)}|}, & |x_j^{(i)}|\in T_S \\ \frac{1}{\varepsilon^{(i)}}, & others\end{cases}$, $g_j(|x_j|+\varepsilon)$ is also a Support Set (T_S). Note that the support set of ISD and RISD is according to first jump rule [9, 11].

5) Y.-B. ZHAO [10](WLP)

$$g_j(|x_j|+\varepsilon)=\frac{1}{p}\sum_{j=1}^{N}(|x_j|+\varepsilon)^p; \quad w_j^{(i+1)}=\frac{1}{(|x_j^{(i)}|+\varepsilon^{(i)})^{1-p}} \quad (6)$$

6) Y.-B. ZHAO [10](NW1)

$$g_j(|x_j|+\varepsilon)=\frac{1}{p}\sum_{j=1}^{N}[|x_j|+\varepsilon+(|x_j|+\varepsilon)^p]$$

$$w_j^{(i+1)}=\frac{p+(|x_j^{(i)}|+\varepsilon^{(i)})^{1-p}}{(|x_j^{(i)}|+\varepsilon^{(i)})^{1-p}[|x_j^{(i)}|+\varepsilon^{(i)}+(|x_j^{(i)}|+\varepsilon^{(i)})^p]} \quad (7)$$

7) Y.-B. ZHAO [10](NW1)

$$g_j(|x_j|+\varepsilon)=\frac{1}{p}\sum_{j=1}^{N}[|x_j|+\varepsilon+(|x_j|+\varepsilon)^q]^p \quad (8)$$

$$w_j^{(i+1)}=\frac{q+(|x_j^{(i)}|+\varepsilon^{(i)})^{1-q}}{(|x_j^{(i)}|+\varepsilon^{(i)})^{1-q}[|x_j^{(i)}|+\varepsilon^{(i)}+(|x_j^{(i)}|+\varepsilon^{(i)})^q]^{1-p}} \quad (9)$$

The existing iteratively reweighted l_1-minimization algorithms are based on a merit function or a support set, from which the weights are derived. Numerical experiments prove that the performance of all algorithms of this family is almost few different [10], which can be seen in numerical experiments section. To further improve the performance of this kind of algorithms, we propose a very simple and efficient algorithm.

3. A SIMPLE METHOD PROPOSED

Based on ISD [11], we proposed a simple algorithm which is outlined as follows.

Input: $\mathbf{A}, \mathbf{y}$, Initialize a set $\Lambda_0 = \varnothing, i = 0$. While $i < i_{\max}$ and the stopping criterion is not met, do

1) Update $\mathbf{x}^{(i)}$ according to

$$\mathbf{x}^{(i)} = \min_{\mathbf{x}} \sum_{j \notin \Lambda_i} \left| x_j \right| \quad subject \quad to \quad \mathbf{Ax} = \mathbf{y} \quad (10)$$

2) Terminate on convergence or when i attains a specified maximum number of iterations $i_{\max}$. Otherwise, increment i .

3) Sort $\{\left| x_j^{(i-1)} \right|\}, j = 1 \cdots N$ in descending order and assign subscripts of the largest $M / 2 \times i / i_{\max}$ (M is the number of measurements) to Λ_i , and the then go to step 1.

The interpretation of the above algorithm is as an iterative reweighted the algorithm with a "0/1" weighting scheme. The nth largest signal coefficients (nth-large signals)are most likely to be identified as nonzero. For the purpose of allowing more sensitivity for identifying the remaining small nonzero signal coefficients (remaining small signals), the influence of nth-large signals should be omitted; while the influence of remaining small signals should be strengthen. Therefore, the weights of nth-large signals are set to '0' in the subsequent iteration, while the weights of remaining small signals are set to '1'. The main reasons are that 1) among the nth-large signals, the probability to be nonzero entries is high, but isn't completely in proportion to their absolute value, 2) among the remaining small signals, the probability to be zero entries is high, but isn't completely is inversely proportional to their absolute value. It is often the case the nth largest signals coefficients include some zero entries and/or the remaining small signals have some nonzero entries. However, numerical results strongly suggest that the new method has a self-corrected capacity.

The advantage of our strategies are 1) no regularization parameter is needed, 2) the weights for all entries are '1' or '0', 3) its performance is better than the existing methods, in terms of both successful probability and complexity of sparse signal recovery.

4. NUMERICAL EXPERIMENTS

There are lots of merit functions or support sets for sparsity, based on which various reweighted l_1-methods can be constructed. This section is to compare these algorithms through numerical experiments. For limited space, we only compare their performances of the five algorithms (WL1FIX, WL2REG, ISD, RISD, Proposed), since the performance of WLP, NW1 and NW2 are almost the same as that of IRL1[10].

4.1 Experimental setting and test platforms

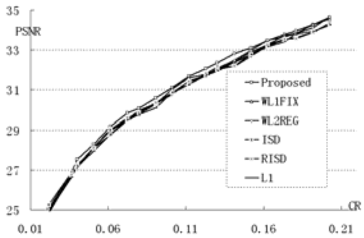

Fig.1 Comparison of the PSNR under the same Compression Ratio for “512×512 Lena” Image for six algorithms

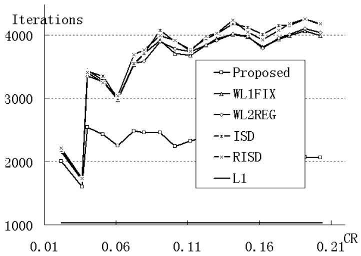

Fig.2：Comparison of the numbers of Iterations for “512×512 Lena” Image for six algorithms

To compare these methods, we use “512×512 Lena” as a test image. For every block, the number of measurements is decided by the following formula.

$$M = \begin{cases} 0.7\times 3\times K, \delta \le 10 \\ 1.4\times 3\times K, others \end{cases} \qquad (11)$$

where δ is the variance of the encoding block, and K is the value of sparsity . For every algorithm, the sparsity K is set from 2 to 20, which are 19 tests in all. Every test is according the following step:

1) Divide the image into 16x16 blocks and then run two dimensions Discreet Cosine Transform (DCT).

2) Each block DCT coefficients convert to 1-D signal and then remove all but the largest K entries from them (signal $\mathbf{x}$).

3) Encode the K -sparsity 256-entries signal by using Measurement matrix $\mathbf{A} \in R^{M\times 256}$ which is random Gaussian matrix generated by MATLAB. The measurements $\mathbf{y}$ is equal to $\mathbf{Ax}$.

4) Signal reconstruct using one of the above five algorithms.

For all tested instances of $\mathbf{Ax} = \mathbf{y}$, the selected iteratively reweighted algorithm was executed, at most 4 iterations, with the same parameters $\varepsilon^{(i)}$ (to be set as [7], only for WL2REG), and the initial point $\mathbf{x}^{(0)} \in R^{256\times 1}$ (the initial value of $\mathbf{x}$ which is set to the solution of the l_1 -minimization). Given a K -sparse solution $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{y}$, the algorithm claims to be successful in finding the K-sparse solution $\mathbf{x}$ if the solution

$\mathbf{x}^{(i)}$ satisfies $\left\|\mathbf{x}^{(i)} - \mathbf{x}\right\|_{l_\infty} \leq 10^{-3}$. To solve these problems, we use CVX, a package for specifying and solving convex programs [20].

5) Each 1-D reconstructed signal convert to 2-D block DCT coefficients.

6) Invert 2-D DCT transform and piece the blocks together a reconstructed image.

4.2 Experimental results

To compare the performance, the compression ratio (CR) is defined as the ratio of measurement numbers and the raw data size.

$$CR = \frac{M_s}{256 \times 256} \qquad (12)$$

here M_s is the number of measurements for all 16x16 blocks according to (12). The compression quality is the peak signal-to-noise ratio (PSNR) is as follows.

$$PSNR = 10\log_{10}\{\frac{512 \times 512 \times 255^2}{\sum_{w=1}^{512}\sum_{h=1}^{512}(x_{w,h}^{rec} - x_{w,h}^{org})^2}\} \qquad (13)$$

where the $x_{w,h}^{rec}$ is the value of the reconstructed pixel of the location (value of the reconstructed pixel of the location (w,h); $x_{w,h}^{org}$ the value of the original pixel of the location (w,h). Clearly, the main computational cost is solving weighted l_1 -minimization problems, so we use the number of iteration to approximately estimate the computation complexity of all the algorithms.

Figure 1 shows that the PSNR of the image reconstructed by using the proposed algorithm is higher than any other existing iteratively reweighted L1-minimization algorithms, closely followed by the WL1FIX, WL2REG, ISD, RISD and L1-minimization, whose average PSNR are 31.09dB, 30.90dB, 30.89dB, 30.82dB, 30.79dB, 30.75dB, respectively.

Figure 2 shows that the computation complexity of the image reconstructed by using the proposed algorithm is lower than any other existing iteratively reweighted L1-minimization algorithms, closely followed by the WL1FIX, WL2REG, ISD, and RISD, whose average PSNR are 2242.63, 3560.74, 3566.73, 3667.11, 3667.58, respectively.

Figure 3and 4 show the CS reconstruction image. There exists block effect in image of figure 3. Image of figure 4 is comparable with the original image (figure 5).

Fig.3. vision quality of CS reconstruction image when K=5, M=17, (PSNR=28.93, CR=0.061)

Fig.4. vision quality of CS reconstruction image when K=10, M=35, (PSNR=32.20, CR=0.13)

Fig. 5. original image

5. CONCLUSION

In summary, we compared both CS reconstruction image quality and computation complexity between the proposed algorithm and the existing reweighted L1-minimization algorithms. The numerical experimental results demonstrate that the proposed algorithm outperforms the others.

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