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Symmetry Analysis of Nonlinear PDE with A "Mathematica" Program SYMMAN

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Abstract

Computer-aided symbolic and graphic computation allows to make significantly easier both theoretical and applied symmetry analysis of PDE. This idea is illustrated by applying a special "Mathematica" package for obtaining conditional symmetries of the nonlinear wave equation $u_t = (u\,u_x)_x$ invariant or partially invariant under its classical Lie symmetries.

1 Introduction

Sophus Lie's approach to classical symmetries reduces their computation to solving overdetermined systems of linear differential equations for coefficients of vector fields of infinitesimal symmetries. Although this kind of computation follows the clear algorithm it yields simple in structure but sometimes enormous in volume calculations with intermediate expressions containing hundreds and thousands of terms for high-order or multidimensional partial differential equations. After contributions of Birkhoff [1] and Ovsiannikov [2], the classical Lie symmetry theory got its renaissance. Remark that Ovsiannikov was first who stimulated the attempts to use symbolic computer-aided symmetry calculations held in Novosibirsk in the sixties [2].

This article deals with non-classical conditional symmetries [3–6]. The algorithm for obtaining infinitesimal conditional symmetries is essentially the same method of Lie's determining equations but these equations are nonlinear and less in number. G.J. Reid's triangularization algorithm [8] or E.L. Mansfeld's and P.A. Clarkson's differential Gröbner basis algorithm [9] programmed by using computer algebra systems can be used for attempts to solve such equations. Here we would like to mention a challenging problem of interplaying the classical and nonclassical symmetries. The matter is that the determining equations for nonclassical symmetries inherit the Lie classical symmetries of the original equations [10]. One can use these classical symmetries for systematical solving of nonclassical determining equations. We demonstrate this method by taking as an example the nonlinear wave equation $u_{tt} = (u\,u_x)_x$ and performing symbolic calculations with "Mathematica" [11].

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2 A "Mathematica" program SYMMAN

An excellent review of programs (more than 16 in number) of symmetry symbolic calculations for Reduce, Macsyma, Maple, Mathematica, Axiom, muMath and other computer algebras is given in [12]. This paper deals with a "Mathematica" program SYMMAN that was designed under the author's supervision at Moscow Institute of Electronics and Mathematics [13]. The program contains five packages that can work separately. The package **Jets.m** is designated for obtaining determining equations for classical and nonclassical symmetries. Moreover, its exported functions provide a wide spectrum of symbolic symmetry calculations. The main task of the package **Involsys.m** is the implementation of generalized Reid's algorithm [8] for reducing (overdetermined) systems of PDEs to involutive forms. Using the package **Res.m** one can partially or completely solve the systems of determining equations. The package **Invarsys.m** does the graphical representation of the invariant solutions of PDEs. The package **Moveline.m** is a subsidiary package for rotating graphs of functions. All five packages can be loaded with the help of the master-package **Master.m**. The program was tested by reobtaining the known symmetry results given in [14].

3 Conditional symmetries of a nonlinear wave equation invariant or partially invariant under the classical Lie symmetries

The nonlinear wave equation

$$u_{tt} - (u\,u_x)_x = 0 \tag{1}$$

admits the four-dimensional Lie algebra $\mathbf{g}$ of its classical infinitesimal symmetries with the generators

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\,\partial_t + x\,\partial_x, \quad X_4 = t\,\partial_t - 2\,u\,\partial_u. \tag{2}$$

We will consider the infinitesimal conditional symmetries of (1) of the form:

$$\mathbf{v} = \partial_t + \xi(t,x)\,\partial_x + (f(t,x)\,u + g(t,x))\;\partial_u. \tag{3}$$

SYMMAN resulted, in particular,with the following relations for the functions $\xi(t,x)$, $f(t,x)$, and $g(t,x)$:

$$f_{xx} = 0, \qquad \xi_{xx} = 4\,f_x, \qquad g = -f\,\xi^2 + 2\,\xi^2\,\xi_x + 2\,\xi\,\xi_t \tag{4}$$

implying that

$$f(t,x) = a(t)\,x + b(t), \qquad \xi(t,x) = 2\,a(t)\,x^2 + c(t)\,x + d(t). \tag{5}$$

If we look for the infinitesimal conditional symmetries $\mathbf{v}$ (3) with the coefficients given by (4) and (5), then one of the determining equations for the functions $a(t)$, $b(t)$, $c(t)$, and $d(t)$ is $a(t) = 0$. After taking this relation into account, we obtain the system of 15 nonlinear ODEs for the functions $b(t)$, $c(t)$, and $d(t)$. By applying SYMMAN for reducing it to passive forms, we managed to obtain the following infinitesimal symmetries:

$$\begin{aligned} \mathbf{v_1} = &\ \partial_t + (c_1\,t + c_2)\,\partial_x + \left(2\,c_1^2\,t + 2\,c_1\,c_2\right)\partial_u, \\ \mathbf{v_2} = &\ (c_1 - (c_2 - 2)\,t)\,\partial_t + 2\,x\,\partial_x + 2\,c_2\,u\,\partial_u, \\ \mathbf{v_3} = &\ 2\,t\,\partial_t + \left(2\,c_1\,t^2 + x\right)\partial_x + 2\,\left(4\,c_1^2\,t^2 - u + 2\,c_1\,x\right)\partial_u, \\ \mathbf{v_4} = &\ t\,\partial_t + c_1\,t^3\,\partial_x + \left(u + 3\,c_1^2\,t^4\right)\partial_u, \\ \mathbf{v_5} = &\ 7\,t\,\partial_t + 3\,x\,\partial_x - 8\,u\,\partial_u, \\ \mathbf{v_6} = &\ 2\,x\,\partial_x + u\,\partial_u. \end{aligned} \tag{6}$$

Vector field $\mathbf{v_1}$ was first obtained in [16], while $\mathbf{v_6}$ in [15].

When reducing overdetermined systems of nonlinear ODEs to passive forms, we had to solve nonlinear algebraic equations. Some expressions for solutions were so cumbersome that could not yield explicit solutions. Sometimes passive forms were systems of ODEs that could not be solved explicitly. That is why we obtained only six vector fields of infinitesimal conditional symmetries. The situation is drastically improved if we restrict ourselves to special classes of the vector fields of the infinitesimal conditional symmetries (3). These classes consist of vector fields invariant or partially invariant under the classical Lie symmetries of equation (1).

Suppose that X is an infinitesimal classical symmetry of equation (1). It was demonstrated in Theorem 4 and Theorem 6 of [10] that if $\mathbf{v} = \tau\,\partial_t + \xi\,\partial_x + \phi\,\partial_u$ is an infinitesimal conditional symmetry of a differential equation and X is an infinitesimal classical symmetry of the same equation, then $\exp(\alpha X)_*(\mathbf{v})$ is also an infinitesimal conditional symmetry of the equation. This assertion implies that the classical Lie symmetry group of the equation generates a symmetry group of the nonclassical determining equations. One can use this induced group for obtaining particular solutions to the nonclassical determining equations, precisely, solutions invariant or partially invariant under the induced group.

The induced vector fields of infinitesimal symmetries are the following ones:

$$\hat{X}_1 = \partial_t, \qquad \hat{X}_2 = \partial_x, \quad \hat{X}_3 = t\partial_t + x\partial_x - \phi\partial_\phi, \quad \hat{X}_4 = t\partial_t - 2u\partial_u - 3\phi\partial_\phi, \tag{7}$$

where ξ and $\phi = f(t,x)\,u + g(t,x)$ are the coefficients of vector field (3). In the case of the Lie algebra $\mathbf{g}$ with the basis given by (7), common methods of obtaining conjugacy classes of subalgebras under the inner automorphisms (see [2]) lead to the following list of representatives of the conjugacy classes (optimal subalgebras):

One-dimensional subalgebras

$$\begin{array}{lll} \mathbf{g_1} = L(X_1), & \mathbf{g_2} = L(X_2), & \mathbf{g_3} = L(X_1 - X_2), \\ \mathbf{g_4} = L(X_1 + X_2), & \mathbf{g_5} = L(X_3), & \mathbf{g_6} = L(X_4), \\ \mathbf{g_7} = L(X_4 + X_2), & \mathbf{g_8} = L(X_4 - X_3), & \mathbf{g_9} = L(X_4 - X_3 + X_1), \\ \mathbf{g_{10}} = L(X_4 - X_3 - X_1), & \mathbf{g_{11}} = L(\zeta X_3 + X_4). & \end{array} \tag{8}$$

Two-dimensional subalgebras

$$\begin{array}{lll} \mathbf{g_{12}} = L(X_4 + \zeta X_3, X_1), & \mathbf{g_{13}} = L(X_4 + \zeta X_3, X_2), & \mathbf{g_{14}} = L(X_4, X_3), \\ \mathbf{g_{15}} = L(X_4 - X_3 + X_1, X_2), & \mathbf{g_{16}} = L(X_4 - X_3, X_2), & \mathbf{g_{17}} = L(X_2, X_1) \\ \mathbf{g_{18}} = L(X_4 + X_2, X_1), & \mathbf{g_{19}} = L(X_4, X_1), & \mathbf{g_{20}} = L(X_4, X_2), \\ \mathbf{g_{21}} = L(X_3, X_1 + X_2), & \mathbf{g_{22}} = L(X_3, X_1 - X_2), & \mathbf{g_{23}} = L(X_3, X_1), \\ \mathbf{g_{24}} = L(X_3, X_2), & \mathbf{g_{25}} = L(X_4 - X_3, X_1), & \end{array} \tag{9}$$

where ζ is a real parameter, $\zeta \neq 0, -1$, L is a symbol of a linear hull.

The following Table 1 contains the infinitesimal conditional symmetries invariant under one-dimensional subalgebras (8).

Table 1. **Invariant conditional symmetries**

$$
\begin{array}{ll}
\mathbf{g_2} & \mathbf{v_{2,1}} = \partial_t + (c_1\,t + c_2)\,\partial_x + \left(2\,c_1^2\,t + 2\,c_1\,c_2\right)\,\partial_u, \\
 & \mathbf{v_{2,2}} = t\,\partial_t + c_1\,t^3\,\partial_x + \left(3\,c_1^2\,t^4 + u\right)\,\partial_u, \\
 & \mathbf{v_{2,3}} = \partial_t + b(t)\,u\,\partial_u \text{ with } b(t) \text{ satisfying } \quad b''(t) + b(t)\,b'(t) - b(t)^3 = 0, \\
\mathbf{g_6} & \mathbf{v_{6,1}} = t^3\,\partial_t + t^2\,(c_1 - x)\,\partial_x + (-6\,c_1^2 + 2\,t^2\,u + 12c_1\,x - 6\,x^2)\,\partial_u, \\
 & \mathbf{v_{6,2}} = t^3\,\partial_t + t^2\,(c_1 + x)\,\partial_x + (c_1^2 - t^2\,u + 2\,c_1\,x + x^2)\,\partial_u, \\
\mathbf{g_7} & \mathbf{v_{7,1}} = t\,\partial_t + u\,\partial_u, \\
\mathbf{g_{11}} & \mathbf{v_{11,1}} = 8\,t\,\partial_t + 9\,x\,\partial_x + 2\,u\,\partial_u, \\
 & \mathbf{v_{11,2}} = 2\,t\,\partial_t + 3\,x\,\partial_x + 2\,u\,\partial_u, \\
 & \mathbf{v_{11,3}} = 2\,t\,\partial_t + \left(2\,c_1\,t^2 + x\right)\,\partial_x + \left(8\,c_1^2\,t^2 - 2\,u + 4\,c_1\,x\right)\,\partial_u, \\
 & \mathbf{v_{11,4}} = 3\,t\,\partial_t + x\,\partial_x - 4\,u\,\partial_u, \\
 & \mathbf{v_{11,5}} = t^3\,\partial_t + \left(c_1\,t^7 - x\,t^2\right)\,\partial_x \\
 & \qquad + \left(4\,c_1^2\,t^{10} + 2\,t^2\,u + 2\,c_1\,t^5\,x - 6\,x^2\right)\,\partial_u, \\
 & \mathbf{v_{11,6}} = t\,\partial_t + 5\,x\,\partial_x + 8\,u\,\partial_u.
\end{array}
$$

Besides invariant solutions, the classical Lie symmetry algebra $\mathbf{g}$ of a system S of partial differential equations provides the method for obtaining partially invariant solutions [2]. The latter are defined relative to at least two-dimensional subalgebras of $\mathbf{g}$. Partially invariant solutions are determined with the aid of partially invariant surface conditions as follows. Suppose that the system S is written for the vector-valued function $u = (u^1(x), \ldots, u^m(x))$ of independent variables $x = (x_1, \ldots, x_n)$. Suppose also that the Lie algebra $\mathbf{g}$ of the infinitesimal symmetries for the system S is a linear hull of its basis vector fields

$$X_j = \xi_{ji}(x,u)\,\partial_{x_i} + \kappa_j^\alpha(x,u)\,\partial_{u^\alpha}, \qquad j = 1, \ldots q. \tag{10}$$

The functions $Q^\alpha(x,u,u_x) = \kappa_j^\alpha(x,u) - u_{x_i}^\alpha\,\xi_{ji}(x,u)$ defined on the space $J^1(R^n \times R^m)$ are called characteristic functions of the basis vector fields (10). The equations of the system

$$\mathrm{rank}||Q_j^\alpha|| \leq \delta, \tag{11}$$

are called partially invariant surface conditions. In (11) δ is the deficiency index. The values of the index δ are restricted by the inequalities $1 \leq \delta \leq \min(q, m-1)$.

There are two unknown functions ξ and ϕ of the independent variables t, x, and u, so there is only one possible value for the deficiency index δ, precisely, $\delta = 1$. Substituting the functions ξ and ϕ into (11) yields the new relation that must be augmented to system (4). The analysis of this new overdetermined system allowed us to obtain the folowing vector fields partially invariant under two-dimensional subalgebras.

Table 2. **Partially invariant conditional symmetries**

$$
\begin{array}{ll}
\mathbf{g_{21}} & \mathbf{v_{21,1}} = 7\,t\,\partial_t + 5\,x\,\partial_x - 4\,u\,\partial_u, \\
 & \mathbf{v_{21,2}} = t^3\,\partial_t - x\,t^2\,\partial_x + \left(2\,u\,t^2 - 6\,x^2\right)\,\partial_u, \\
 & \mathbf{v_{21,3}} = t^3\,\partial_t + x\,t^2\,\partial_x + \left(-u\,t^2 + x^2\right)\,\partial_u, \\
\mathbf{g_{24}} & \mathbf{v_{24,1}} = 4\,t^3\,\partial_t + 2\,(5 - \sqrt{13})\,t^2\,x\,\partial_x + 4\,t^2\,u\,(3 - \sqrt{13})\,\partial_u, \\
 & \mathbf{v_{24,2}} = 4\,t^3\,\partial_t + 2(5 + \sqrt{13})\,t^2\,x\,\partial_x + 4\,t^2\,u\,(3 + \sqrt{13})\,\partial_u.
\end{array}
$$

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