Formal Linearization and Exact Solutions of Some Nonlinear Partial Differential Equations

V.A. BAIKOV*, K.R. KHUSNUTDINOVA**

*Ufa State Aviation Technical University,
12 K.Marx Str., 450000 Ufa, Russia
E-mail: baikov@mathem.uaicnit.bashkiria.su

**Bashkir State University,
32 Frunze Str., 450074 Ufa, Russia
E-mail: karima@fractal.bashkiria.su

Abstract

An efficient method for constructing of particular solutions of some nonlinear partial differential equations is introduced. The method can be applied to nonintegrable equations as well as to integrable ones. Examples include multisoliton and periodic solutions of the famous integrable evolution equation (KdV) and the new solutions, describing interaction of solitary waves of nonintegrable equation.

1   Introduction

In recent years there was interest in constructing solutions of nonlinear partial differential equations in the form of infinite series. The direct linearization of certain famous integrable nonlinear equations was carried out in [1]. Solutions of the KdV equation were connected with solutions of the Hopf equation by using formal series in [2] (the Hopf equation can be linearized with the help of the "hodograph" transformation). Convergent exponential series were used in papers [3]–[8] for constructing solutions of the Boltzmann equations. The possibility to use such series for some other equations was discussed in [4]. Fourier series were applied for constructing solutions of perturbed KdV equation in [9]. Exponential series were used also for investigating nonlinear elliptic equations [10]. Some other references can be found in the cited papers.

In this paper we consider the class of equations and systems containing arbitrary linear differential operators with constant coefficients and arbitrary nonlinear analytic functions of dependent variables and their derivatives up to some finite order in assumption that these equations possess a constant solution. In contrast to the cited papers [1] and [2], we do not look for transformation connecting solutions of the given equation with an arbitrary solution of some other equation. Our method is based on formal linearization of a nonlinear partial differential equation to the system of linear ordinary differential equations, describing some finite-dimensional subspace of the space of solutions of the linearized equation. It allows us to develop a very simple technique of finding the linearizing transformation and to apply the method to nonintegrable equations as well as to integrable ones. Solutions have the form of exponential or Fourier series.

Let us note that the similar approach with the different technique (solutions are constructed in the form of exponential series in x with coefficients, depending on t and determined by the system of ordinary differential equations, which can be solved recursively) was independently developed in [11] for the wide class of evolution equations and in this case the convergence of constructed exponential series was investigated [11].

2 The method of formal linearization

Let us consider equations of the following form

$$\hat{L}(D_t, D_x)u(t, x) = N[u], \tag{1}$$

where

$$\hat{L}(D_t, D_x) \equiv \sum_{k=0}^{K} \sum_{m=0}^{M} l_{km} D_t^k D_x^m \tag{2}$$

is a linear differential operator with constant coefficients and

$$N[u] \equiv N(u, u_1, u_2, ..., u_p), \quad u_p = \frac{\partial^{p_1+p_2}u}{\partial t^{p_1}\partial x^{p_2}}, \quad p = (p_1, p_2),$$

is an arbitrary analytic function of u and of its derivatives up to some finite order p. We suppose that Eq.(1) possesses the constant solution. Without loss of generality we assume that

$$N[0] = 0, \quad \frac{\partial N[0]}{\partial u} = 0, \quad \frac{\partial N[0]}{\partial u_1} = 0, \quad ..., \quad \frac{\partial N[0]}{\partial u_p} = 0.$$

We consider Eq.(1) in connection with the equation linearized near a zero solution:

$$\hat{L}(D_t, D_x)w(t, x) = 0. \tag{3}$$

Let L be the vector space of solutions of Eq.(3) and $P^N \subset L$ be the N-dimensional subspace with the basis

$$w_i = W_i \exp(\alpha_i \xi_i), \quad \xi_i = x - s_i t, \quad i = \overline{1, N}.$$

Here s_i and W_i are some constants. The constants $\alpha_i = \alpha_i(s_i)$ are assumed to satisfy the dispersion relation

$$\hat{L}(-\alpha_i s_i, \alpha_i) = 0.$$

The subspace $P^N = \left\{ \sum_{i=1}^{N} C_i w_i | C_i = \text{const} \right\}$ is specified by the system of N linear ordinary differential equations

$$\frac{dw_i}{d\xi_i} = \alpha_i w_i, \quad i = \overline{1, N}.$$

We use the following notation:

$$w_{(N)}^{\delta} \equiv w_1^{\delta_1} w_2^{\delta_2}...w_N^{\delta_N}, \quad \delta = (\delta_1, \delta_2, ..., \delta_N), \quad |\delta| = \sum_{i=1}^{N} \delta_i.$$

It is obvious that the monomials $w^\delta_{(N)}$ are the eigenfunctions of the operator (2):

$$\hat{L}(D_t, D_x)w^\delta_{(N)} = \lambda_\delta w^\delta_{(N)}$$

with the eigenvalues

$$\lambda_\delta = \sum_{k=0}^{K} \sum_{m=0}^{M} l_{km}\Big(-\sum_{i=1}^{N} \alpha_i s_i \delta_i\Big)^k \Big(\sum_{i=1}^{N} \alpha_i \delta_i\Big)^m.$$

Theorem 1. *If $\lambda_\delta \neq 0$ for every multiindex δ with positive integer components $\delta_i \in Z_+$, $i = \overline{1, N}$, satisfying the condition $|\delta| \neq 0, 1$, then Eq.(1) possesses solutions connected with solutions from P^N by the formal transformation*

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2, ..., w_N), \qquad (4)$$

where

$$\phi_n = \sum_{|\delta|=n} (A_n)_\delta w^\delta_{(N)} \qquad (5)$$

are homogeneous polynomials of degree n in the variables w_i. This transformation is unique (for the first term $\phi_1 \in P^N$ fixed).

Remark 1. Here ε is the grading parameter, finally we can put $\varepsilon = 1$.

The proof of the theorem is constructive. Substituting (4) into (1), expanding $N[u]$ into the power series in ε, and then collecting equal powers of ε, we obtain the determining equations for the functions ϕ_n and show that if $\lambda_\delta \neq 0$, then these equations possess the solution (5) with the coefficients $(A_n)_\delta$ uniquely determined through the coefficients $(A_1)_\delta$ by the recursion relation. Thus, the theorem gives us the method for constructing particular solutions of Eq.(1).

This result can be generalized for the systems of the form

$$\sum_{j=1}^{n} \hat{L}^i_j(D_t, D_x)u^j(t, x) = N^i[u], \quad i = \overline{1, n}, \qquad (6)$$

where

$$\hat{L}^i_j(D_t, D_x) \equiv \sum_{k=0}^{K} \sum_{m=0}^{M} (l^i_j)_{km} D_t^k D_x^m \qquad (7)$$

are linear differential operators with constant coefficients and $N^i[u]$ are arbitrary analytic functions of $u^j, j = \overline{1, n}$, and their derivatives up to some finite order p. We suppose again that the system (6) possesses a constant solution. There is no loss of generality in assuming that

$$N^i[0] = 0, \quad \frac{\partial N^i[0]}{\partial u^j} = 0, \quad \frac{\partial N^i[0]}{\partial u^j_1} = 0, \quad ..., \quad \frac{\partial N^i[0]}{\partial u^j_p} = 0; \quad i, j = \overline{1, n}.$$

The system linearized near a zero solution and corresponding to (6) has the form

$$\sum_{j=1}^{n} \hat{L}_j^i(D_t, D_x) w^j(t, x) = 0. \tag{8}$$

Now the subspace P^N of solutions of (8) is generated by the vector functions

$$w_l = (W_l^1, W_l^2, ..., W_l^n) \exp(\alpha_l \xi_l), \quad \xi_l = x - s_l t, \quad l = \overline{1, N}$$

with some constants s_l and W_l^j. Here the constants $\alpha_l = \alpha_l(s_l)$ are assumed to satisfy the dispersion relation

$$\det [\hat{L}_j^i(-\alpha_l s_l, \alpha_l)] = 0$$

and the constants $W_l^j = W_l^j(s_l)$ are assumed to satisfy the system of linear algebraic equations

$$\sum_{j=1}^{n} \hat{L}_j^i(-\alpha_l s_l, \alpha_l) W_l^j = 0, \quad i = \overline{1, n}.$$

The subspace P^N is specified by the system of N linear ordinary differential equations

$$\frac{dw_l^1}{d\xi_l} = \alpha_l w_l^1, \quad l = \overline{1, N}$$

and the set of $(n-1)N$ algebraic relations $w_l^j = W_l^j w_l^1, \quad j = \overline{2, n}, \quad l = \overline{1, N}.$

All constructions can be repeated. Let us use the notation $(\lambda_j^i)_\delta$ for the eigenvalues of the monomials

$$(w_{(N)}^1)^\delta \equiv (w_1^1)^{\delta_1} (w_2^1)^{\delta_2} ... (w_N^1)^{\delta_N}$$

under action of the operators (7):

$$\hat{L}_j^i(D_t, D_x)(w_{(N)}^1)^\delta = (\lambda_j^i)_\delta (w_{(N)}^1)^\delta.$$

Here

$$(\lambda_j^i)_\delta = \sum_{k=0}^{K} \sum_{m=0}^{M} (l_j^i)_{km} \left(-\sum_{l=1}^{N} \alpha_l s_l \delta_l \right)^k \left(\sum_{l=1}^{N} \alpha_l \delta_l \right)^m.$$

Theorem 2. *If* $\det \ [(\lambda_j^i)_\delta] \neq 0$ *for every multiindex* δ *with positive integer components* $\delta_l \in Z_+$, $l = \overline{1, N}$, *satisfying the condition* $|\delta| \neq 0, 1$, *then the system (6) possesses solutions connected with the solutions from* P^N *by the formal transformation*

$$u^j = \sum_{k=1}^{\infty} \varepsilon^k \phi_k^j(w_1^1, w_2^1, ..., w_N^1), \quad j = \overline{1, n},$$

where

$$\phi_k^j = \sum_{|\delta|=k} (A_k^j)_\delta (w_{(N)}^1)^\delta$$

are homogeneous polynomials of degree k in the variables w_l^1. This transformation is unique (for the first term $\phi_1 \in P^N$ fixed).

Remark 2. Although the conditions $\lambda_\delta \neq 0$ and $\det[(\lambda_j^i)_\delta] \neq 0$ seem to be very restrictive, they are usually fulfilled in some open domain of the parameter space, as it is shown by the examples.

3 Examples

3.1 The KdV equation (multisoliton solutions)

Let us consider the KdV equation

$$\hat{L}(D_t, D_x)u(t,x) = -6uu_x, \quad \hat{L}(D_t, D_x) = D_t + D_x^3. \tag{9}$$

For simplicity we look for a solution of (9) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \tag{10}$$

where

$$w_i = W_i \exp[\sqrt{s_i}(x - s_i t)], \quad i = 1, 2$$

is the basis of the subspace $P^2 \subset L$ (let s_i and W_i be some real constants).

Substituting (10) into (9) and collecting equal powers of ε we obtain the determining equations for the functions ϕ_n as follows

$$\hat{L}\phi_1 = 0, \quad \hat{L}\phi_n = -6\sum_{k=1}^{n-1} \phi_k D_x \phi_{n-k}, \quad n \geq 2.$$

These equations possess the solution $\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(2)}^\delta, \quad \delta = (\delta_1, \delta_2)$, which can be rewritten in this case in the following form

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k} \quad (\phi_1 \in P^2).$$

The coefficients A_k^n can be found through A_0^1 and A_1^1 (we can assume that either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$) by the recursion relation

$$A_k^n = -\frac{6}{\lambda_{(k,n-k)}} \sum_{l=1}^{n-1} \sum_{m=0}^{n-l} [\sqrt{s_1}m + \sqrt{s_2}(n-l-m)]A_{k-m}^l A_m^{n-l}, \tag{11}$$

$$n \geq 2, \quad 0 \leq k \leq n; \quad A_k^n = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n,$$

$$\lambda_{(k,n-k)} = s_1\sqrt{s_1}k(k^2 - 1) + s_2\sqrt{s_2}(n-k)[(n-k)^2 - 1] + 3\sqrt{s_1 s_2}k(n-k)[\sqrt{s_1}k + \sqrt{s_2}(n-k)].$$

If $s_1 > 0$ and $s_2 > 0$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in Z_+$, $n \geq 2$, $0 \leq k \leq n$.

The constructed solution is an expansion of a 2-soliton solution, which can be written as

$$u = 2D_x^2 \, \ln \left(1 + w_1 + w_2 + \left(\frac{\sqrt{s_1} - \sqrt{s_2}}{\sqrt{s_1} + \sqrt{s_2}} \right)^2 w_1 w_2 \right). \tag{12}$$

Indeed, expanding (12) into power series in w_1 and w_2 and substituting this expansion into (9), we obtain the recursion relation (11) for the coefficients.

If $A_0^1 = 0$, then $\phi_1 \in P^1$ and we get from (10) the expansion for a 1-soliton solution. For obtaining the N-soliton solution, we must take $\phi_1 \in P^N$.

3.2   Nonintegrable equation (solutions, describing interaction of solitary waves)

Let us consider the equation

$$\hat{L}(D_t, D_x)u(t, x) = u^3, \quad \hat{L}(D_t, D_x) = D_t^2 - D_x^2 + 1. \tag{13}$$

It is known [12] that this equation is nonintegrable and its solitary wave solutions are not solitons. We can construct analytic solutions describing interaction of these waves with the help of our method.

Constant solutions of (13) are $u = 0$ and $u = \pm 1$.

The equation linearized near a zero solution has the form $\hat{L}w = 0$ and the space of its solutions contains the subspace P^2 with the basis

$$w_i = W_i \exp \frac{x - s_i t}{\sqrt{1 - s_i^2}}, \quad i = 1, 2.$$

We look for solutions of (13) in the form (10) and obtain the determining equations as follows

$$\hat{L}\phi_1 = 0, \quad \hat{L}\phi_2 = 0, \quad \hat{L}\phi_n = \sum_{k=2}^{n-1} \phi_{n-k} \sum_{l=1}^{k-1} \phi_l \phi_{k-l}, \quad n \geq 3.$$

These equations possess the solution

$$\phi_{2p+1} = \sum_{n=0}^{2p+1} A_n^p w_1^n w_2^{2p+1-n}, \quad \phi_{2p+2} = 0, \quad p \geq 0,$$

where

$$A_n^p = \frac{1}{\lambda_{(n,2p+1-n)}} \sum_{m=0}^{p-1} \sum_{r=o}^{p-m-1} \sum_{k=0}^{2m+1} \sum_{l=o}^{2r+1} A_k^m A_l^r A_{n-k-l}^{p-m-r-1},$$

$$p \geq 1, \quad 0 \leq n \leq 2p+1; \quad A_n^p = 0 \quad \text{if} \quad n < 0 \quad \text{or} \quad n > 2p+1;$$

$$\lambda_{(n,2p+1-n)} = 1 - n^2 - (2p+1-n)^2 - 2\frac{1 - s_1 s_2}{\sqrt{(1 - s_1^2)(1 - s_2^2)}} n(2p+1-n).$$

Here either $A_0^0 = A_1^0 = 1$ (in this case $A_n^p = A_{2p+1-n}^p$) or $A_0^0 = 0, A_1^0 = 1$.

If $|s_1| \leq 1$ and $|s_2| \leq 1$, then $\lambda_{(n,2p+1-n)} \neq 0$ for every pair $(n, 2p + 1 - n)$ with $n, p \in Z_+,$ $p \geq 1,$ $0 \leq n \leq 2p + 1.$

If $A_0^0 = 0$, then we obtain

$$u = \sum_{p=0}^{\infty} (-\frac{1}{8})^p (\varepsilon w_1)^{2p+1} = \frac{\varepsilon w_1}{1 + \frac{1}{8}(\varepsilon w_1)^2} = \frac{2\sqrt{2}w}{1 + w^2},$$

where $w = \varepsilon w_1/2\sqrt{2}$. In (t, x)-variables we have

$$u = \pm\sqrt{2} \ \text{sech} \ \frac{x - st + x_0}{\sqrt{1 - s^2}}.$$

These are solitary wave solutions. Solitary waves of this kind can move with velocities satisfying the condition $|s| \leq 1$.

Another possibility to construct solutions is to linearize (13) near the solution $u = 1$. The change of variables $u = 1 + v$ leads to the equation

$$\hat{M}(D_t, D_x)v(t, x) = 3v^2 + v^3, \quad \hat{M}(D_t, D_x) = D_t^2 - D_x^2 - 2,$$

which should be linearized near a zero solution. In this case, the subspace P^2 is generated by the functions

$$w_i = W_i \exp\left[\sqrt{\frac{2}{s_i^2 - 1}}(x - s_i t)\right], \quad i = 1, 2.$$

Our procedure gives the solution

$$v = \sum_{n=1}^{\infty} \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k},$$

$$A_k^n = \frac{1}{\tilde{\lambda}_{(k,n-k)}} \left\{ 3\sum_{l=1}^{n-1}\sum_{m=o}^{l} A_m^l A_{k-m}^{n-l} + \sum_{l=2}^{n-1}\sum_{m=1}^{l-1}\sum_{p=0}^{n-l}\sum_{q=0}^{m} A_p^{n-l} A_q^m A_{k-p-q}^{l-m} \right\},$$

$$n \geq 2, \quad 0 \leq k \leq n; \quad A_k^n = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n;$$

$$\tilde{\lambda}_{(k,n-k)} = 2\left[k^2 + (n-k)^2 + 2\frac{s_1 s_2 - 1}{\sqrt{(s_1^2 - 1)(s_2^2 - 1)}}k(n-k) - 1\right].$$

Here either $A_0^1 = A_1^1 = 1$ (in this case $A_k^n = A_{n-k}^n$) or $A_0^1 = 0, A_1^1 = 1$.

If $|s_1| > 1$ and $|s_2| > 1$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n - k)$ with $n, k \in Z_+,$ $n \geq 2,$ $0 \leq k \leq n.$

If $A_0^1 = 0$, then we get

$$v = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} (\varepsilon w_1)^n = \frac{\varepsilon w_1}{1 - \frac{1}{2}\varepsilon w_1} = \frac{2w}{1 - w},$$

where $w = \varepsilon w_1/2$. Thus, in (t, x)-variables we have

$$u = 1 + v = \pm\tanh\frac{x - st + x_0}{\sqrt{2(s^2 - 1)}}.$$

These are the kinks which can move with velocities satisfying the condition $|s| > 1$.

Linearization near the solution $u = -1$ does not give new solutions.

4   Concluding remarks

It should be noted that the method of formal linearization introduced in this paper can be developed by means of linearization of a nonlinear partial differential equation near its nonconstant solutions. For example, the KdV equation (9) possesses the exact singular solution

$$u = 2s\frac{w}{(1+w)^2}, \quad w = W\exp[\sqrt{s}(x - st)], \quad s < 0.$$

Linearizing (9) near this solution, we obtain the expansion for the well-known cnoidal wave solution. The expansion contains not only positive but also negative powers of w. Thus, in this case we deal with Fourier series.

Acknowledgement

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