Oriented bicyclic graphs whose skew spectral radius does not exceed 2

Jia-Hui Ji · Guang-Hui Xu

Abstract Let $S(G^\sigma)$ be the skew-adjacency matrix of the oriented graph G^σ of order n and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be all eigenvalues of $S(G^\sigma)$. The skew-spectral radius $\rho_s(G^\sigma)$ of G^σ is defined as $\max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}$. In this paper, we investigate oriented bicyclic graphs whose skew-spectral radius does not exceed 2.

Key words: Oriented bicyclic graph · Skew-adjacency matrix · Skew-spectral radius.

1 Introduction

Let G be a simple graph with n vertices. The *adjacency matrix* $A = A(G)$ is the symmetric matrix $[a_{ij}]_{n\times n}$ where $a_{ij} = a_{ji} = 1$ if v_iv_j is an edge of G, otherwise $a_{ij} = a_{ji} = 0$. We call $det(\lambda I - A)$ the *characteristic polynomial* of G, denoted by $\phi(G;\lambda)$ (or abbreviated to $\phi(G)$). Since A is symmetric, its eigenvalues $\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)$ are real, and we assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. We call $\rho(G) = \lambda_1(G)$ the *adjacency spectral radius* of G.

The graph obtained from a simple undirected graph by assigning an orientation to each of its edges is referred as the *oriented graph*. Let G^σ be an oriented graph with vertex set $\{v_1, v_2, \cdots, v_n\}$ and edge set $E(G^\sigma)$. The *skew-adjacency matrix* $S = S(G^\sigma) = [s_{ij}]_{n\times n}$ related to G^σ is defined as:

$$s_{ij} = \begin{cases} \mathbb{i}, & \text{if there exists an edge with tail } v_i \text{ and head } v_j; \\ -\mathbb{i}, & \text{if there exists an edge with head } v_i \text{ and tail } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

Where $\mathbb{i} = \sqrt{-1}$ (Note that the definition is slightly different from the one of the normal skew-adjacency matrix given by Adiga *et al.* [1]). Since $S(G^\sigma)$ is an Hermitian

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matrix, the eigenvalues $\lambda_1(G^\sigma), \lambda_2(G^\sigma), \cdots, \lambda_n(G^\sigma)$ of $S(G^\sigma)$ are all real numbers and thus can be arranged non-increase as $\lambda_1(G^\sigma) \geq \lambda_2(G^\sigma) \geq \cdots \geq \lambda_n(G^\sigma)$. The *skew-spectral radius* and the *skew-characteristic polynomial* of G^σ are defined respectively as $\rho_s(G^\sigma) = \lambda_1(G^\sigma)$ and $\phi(G^\sigma;\lambda) = det(\lambda I_n - S(G^\sigma))$.

In this paper, we will investigate oriented bicyclic graphs whose skew-spectral radius does not exceed 2.

2 Preliminaries

Let $G = (V,E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and $e \in E(G)$. Denote by $G - e$ the graph obtained from G by deleting the edge e and by $G - v$ the graph obtained from G by removing the vertex v together with all edges incident to it. For a nonempty subset W of $V(G)$, the subgraph with vertex set W and edge set consisting of those pairs of vertices that are edges in G is called an *induced subgraph* of G.

In terms of defining subgraph, degree and matching of an oriented graph, we focus only on its underlying graph. Certainly, each subgraph of an oriented graph is also referred as an oriented graph and preserve the orientation of each edge. Denote by C_n, $K_{1,n-1}$ and P_n the cycle, the star and the path on n vertices, respectively.

Recall that the skew adjacency matrix $S(G^\sigma)$ of any oriented graph G^σ is Hermitian, then the well known interlacing Theorem for Hermitian matrices applies equally well to oriented graphs; see for example Theorem 4.3.8 of [5].

Lemma 1. *Let G^σ be an arbitrary oriented graph on n vertices and $V' \subseteq V(G)$. Suppose that $|V'| = k$. Then*

$$\lambda_i(G^\sigma) \geq \lambda_i(G^\sigma - V') \geq \lambda_{i+k}(G^\sigma) \;\; for \;\; i = 1, 2, \cdots, n-k. \tag{2.1}$$

Let G^σ be an oriented graph and $C = v_1 v_2 \cdots v_k v_1 (k \geq 3)$ be a cycle of G, where v_j adjacent to v_{j+1} for $j = 1, 2 \cdots, k-1$ and v_1 adjacent to v_k. Let also $S(G^\sigma) = [s_{ij}]_{n\times n}$ be the skew-adjacency matrix of G^σ whose first k rows and columns correspond the vertices $v_1, v_2, \cdots, v_k$. The sign of the cycle C^σ, denoted by $sgn(C^\sigma)$, is defined by

$$sgn(C^\sigma) = s_{1,2} s_{2,3} \cdots s_{k-1,k} s_{k,1}.$$

Let $\bar{C} = v_1 v_k \cdots v_2 v_1$ be the cycle by inverting the order of the vertices along the cycle C. Then one can verify that

$$sgn(\bar{C}^\sigma) = \begin{cases} -sgn(C^\sigma), & \text{if } k \text{ is odd;} \\ sgn(C^\sigma), & \text{if } k \text{ is even.} \end{cases}$$

Moreover, $sgn(C^\sigma)$ is either 1 or -1 if the length of C is even; and $sgn(C^\sigma)$ is either $\mathtt{i}$ or $-\mathtt{i}$ if the length of C is odd. For an even cycle, we simply refer it as a positive cycle or a negative cycle according to its sign.

Lemma 2. ([9]) *Let G^σ be an arbitrary connected oriented graph. Denote by $\rho(G)$ the (adjacency) spectral radius of G. Then*

$$\rho_s(G^\sigma) \leq \rho(G).$$

with equality if and only if G is bipartite and each cycle of G is a positive even cycle.

Lemma 3. ([9]) *Let G^σ be a connected oriented graph. Suppose that each even cycle of G is positive. Then*
(a) $\rho_s(G^\sigma) > \rho_s(G^\sigma - u)$ for any $u \in G$;
(b) $\rho_s(G^\sigma) > \rho_s(G^\sigma - e)$ for any $e \in G$.

Lemma 4. ([6, 9]) *Let G^σ be an oriented graph and $\phi(G^\sigma, \lambda)$ be its skew-characteristic polynomial. Then*

(a) $\phi(G^\sigma,\lambda) = \lambda\phi(G^\sigma - u,\lambda) - \sum_{v\in N(u)} \phi(G^\sigma - u - v,\lambda) - 2\sum_{u\in C} sgn(C)\phi(G^\sigma - C,\lambda),$

where the first summation is over all the vertices in $N(u)$ and the second summation is over all even cycles of G containing the vertex u.

(b) $\phi(G^\sigma,\lambda) = \phi(G^\sigma - e,\lambda) - \phi(G^\sigma - u - v,\lambda) - 2\sum_{(u,v)\in C} sgn(C)\phi(G^\sigma - C,\lambda),$

where $e = (u,v)$ and the summation is over all even cycles of G containing the edge e and $sgn(C)$ denotes the sign of the even cycle C.

Lemma 5. ([4]) Let G^σ be an oriented graph and $\phi(G^\sigma, \lambda)$ be its skew-characteristic polynomial. Then

$$\frac{d}{d\lambda}\phi(G^\sigma,\lambda) = \sum_{v\in V(G)} \phi(G^\sigma - v,\lambda),$$

where $\frac{d}{d\lambda}\phi(G^\sigma,\lambda)$ denotes the derivative of $\phi(G^\sigma,\lambda)$.

Finally, we introduce a class of undirected graphs that will be often mentioned in this manuscript.

Denote by $P_{l_1,l_2,\cdots,l_k}$ a pathlike oriented graph, which is defined as follows: we first draw $k(\geq 2)$ oriented paths $P_1, P_2, \cdots, P_k$ of order $l_1, l_2, \cdots, l_k$ respectively along a line and put two isolated vertices between each pair of those paths, then add oriented edges between the two isolated vertices and the nearest end vertices of such a pair path such that the four new adding oriented edges forms a negative oriented quadrangle, where $l_1, l_k \geq 0$ and $l_i \geq 1$ for $i = 2, 3, \cdots, k-1$. Then $P_{l_1,l_2,\cdots,l_k}$ contains $\sum_{i=1}^{k} l_i + 2k - 2$ vertices. Notices that if $l_i = 1 (i = 2, 3, \cdots, k-1)$, then the two end vertices of the path P_i are referred as overlap; if $l_1 = 0$ ($l_k = 0$), then the left (right)

of the graph $P_{l_1,l_2,\cdots,l_k}$ has only two pendent vertices. Obviously, $P_{1,0} = K_{1,2}$, the star of order 3, and $P_{1,1}$ is the negative oriented quadrangle. See Figure 2.1 for three pathlike oriented graphs $P_{0,n-4,0}$, $P_{1,n-6,1}$ and $P_{0,l_1,2,l_2,2}(l_1, l_2 \geq 1)$.

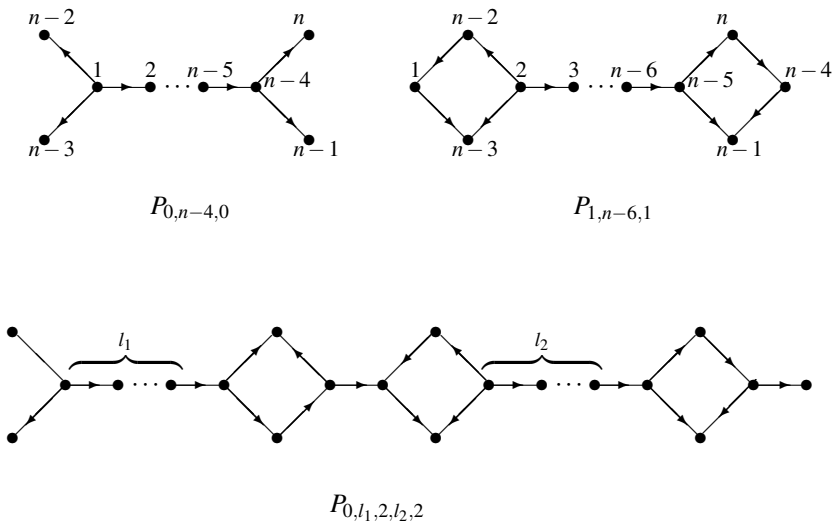

Figure 2.1. Three pathlike graphs $P_{0,n-4,0}$, $P_{1,n-6,1}$ and $P_{0,l_1,2,l_2,2}(l_1, l_2 \geq 1)$.

For convenience of discussion, we call the oriented graph $P_{l_1,l_2,\cdots,l_k}$ as a $k-$path oriented graph, according to k, and call the vertex v as an inner vertex if v is contained in some path among $P_1, P_2, \cdots, P_k$ mentioned as above, and an outer vertex otherwise. Then we have

Observation 1. Let $P_{l_1,l_2,\cdots,l_k}(k \geq 3)$ be a pathlike oriented graph defined as above and $v \in P_{l_1,l_2,\cdots,l_k}$. Then
(a) $P_{l_1,l_2,\cdots,l_k} - v$ is of $(k-1)-$path oriented graphs if v is an outer vertex;
(b) $P_{0,l_2,\cdots,l_{k-1},0} - v$ is the union of two isolated vertices and a $(k-1)-$path oriented graph if $P_{0,l_2,\cdots,l_{k-1},0} \neq P_{0,1,0}$ and v is the left (or right) inner vertex of P_2 (or P_{k-1});
(c) $P_{0,l_2,\cdots,l_{k-1},0} - v$ is the union of an $i-$path oriented graph and a $(k-i+1)-$path oriented graph if $P_{0,l_2,\cdots,l_{k-1},0} \neq P_{0,1,0}$ and v of P_i, except the two inner vertices mentioned as (b), for $i = 2, 3, \cdots, k-1$.

Lemma 6. *Let $P_{l,n-l-2}$ be the pathlike oriented graph on $n(\geq 3)$ vertices described as above. Then*

$$\rho_s(P_{l,n-l-2}) < 2.$$

Proof. We first show by induction on n that

$$\phi(P_{l,n-l-2}, 2) = 4. \tag{2.2}$$

Let $l \geq n-l-2$. Then there has exactly one pathlike graph if $n=3$, namely, $P_{1,0} = P_3$, and exactly two pathlike graphs if $n=4$, namely, $P_{1,1}$ and $P_{2,0} = K_{1,3}$. By a direct calculation, we have

$$\phi(P_{1,0},2) = \phi(P_{1,1},2) = \phi(P_{2,0},2) = 4.$$

Suppose now that $n \geq 5$ and the result is true for the order no more than $n-1$. For $P_{l,n-l-2}$, applying Lemma 4 to the left pendent vertex of $P_{l,n-l-2}$, we have

$$\phi(P_{l,n-l-2},\lambda) = \lambda\phi(P_{l-1,n-l-2},\lambda) - \phi(P_{l-2,n-l-2},\lambda).$$

Then $\phi(P_{l,n-l-2},2) = 4$ by induction hypothesis and thus the result follows.

Let now v be an arbitrary outer vertices of $P_{l,n-l-2}$. Then $P_{l,n-l-2} - v = P_{n-1}$, a path of order $n-1$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1}$ be all eigenvalues of $P_{l,n-l-2}$ and P_{n-1}, respectively. By Lemma 1 and the fact that $\bar{\lambda}_1 < 2$, we have $\lambda_2 \leq \bar{\lambda}_1 < 2$. On the other hand, we have

$$\phi(P_{l,n-l-2},\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i).$$

Consequently, $\lambda_1 < 2$ and thus $\rho_s(P_{l,n-l-2}) < 2$ by Eq.(2.2). Thus the proof is complete. ■

Lemma 7. *Let $P_{l_1,l_2,\cdots,l_k}(k \geq 2)$ be a pathlike oriented graph described as above. Then*

$$\rho_s(P_{l_1,l_2,\cdots,l_k}) \leq 2.$$

Moreover, 2 is an eigenvalue of $P_{l_1,l_2,\cdots,l_k}$ with multiplicity $k-2$.

Proof. We prove the result by induction on k. By Lemma 6, the result follows for $k=2$. Suppose that the result is true for any graph $P_{l_1,l_2,\cdots,l_k}$ with $k \leq t$ for some fixed integer t and $t \geq 2$. For $k = t+1$, we first show that 2 is an eigenvalue of $P_{l_1,l_2,\cdots,l_{t+1}}$ with multiplicity at least $t-1$.

For $P_{0,l_2,\cdots,l_t,0}$, the oriented graph whose two ends contain respectively two pendent vertices, applying Lemma 5, we have

$$\frac{d}{d\lambda}\phi(P_{0,l_2,\cdots,l_t,0},\lambda) = \sum_{v}\phi(P_{0,l_2,\cdots,l_t,0} - v,\lambda).$$

By Observation 1, $P_{0,l_2,\cdots,l_t,0} - v$ is either a $t-$path oriented graph, or the union of two isolated vertices and a $t-$path oriented graph, or the union of an $i-$path oriented graph and an $(t+2-i)-$path oriented graph for $i = 2,3,\cdots,t$. By induction hypothesis, 2 is an eigenvalue of any $p-$path oriented graph with multiplicity $p-2$ for $2 \leq p \leq t$. Then $\phi(P_{0,l_2,\cdots,l_t,0} - v,\lambda)$ contains the factor $(\lambda-2)^{t-2}$ for any $v \in P_{0,l_2,\cdots,l_t,0}$ and thus $\frac{d}{d\lambda}\phi(P_{0,l_2,\cdots,l_t,0},\lambda)$ contains the factor $(\lambda-2)^{t-2}$. Consequently, combining with lemma 4, 2 is an eigenvalue of $P_{0,l_2,\cdots,l_t,0}$ with multiplicity at least $t-1$.

For $P_{l_1,l_2,\cdots,l_t,0}$, we prove the result by induction on l_1. By the discussion above, the result holds for $l_1 = 0$. For $P_{1,l_2,\cdots,l_t,0}$, applying Lemma 4 to the unique vertex of P_1, we have

$$\phi(P_{1,l_2,\cdots,l_t,0},\lambda) = \lambda\phi(P_{0,l_2,\cdots,l_t,0},\lambda) - 2\phi(P_{l_2+1,\cdots,l_t,0},\lambda) - 2\phi(P_{l_2-1,\cdots,l_t,0},\lambda).$$

The result follows since all of $\phi(P_{0,l_2,\cdots,l_t,0},\lambda)$, $\phi(P_{l_2+1,\cdots,l_t,0},\lambda)$ and $\phi(P_{l_2-1,\cdots,l_t,0},\lambda)$ contain the factor $(\lambda-2)^{t-1}$ by the proof above and the induction hypothesis. Suppose that the result is true for any graph $P_{l_1,l_2,\cdots,l_t,0}$ with small l_1. For $P_{l_1+1,l_2,\cdots,l_t,0}$, applying Lemma 4 to the right pendent vertex of $P_{l_1+1,l_2,\cdots,l_t,0}$, we have

$$\phi(P_{l_1+1,l_2,\cdots,l_t,0},\lambda) = \lambda\phi(P_{l_1,l_2,\cdots,l_t,0},\lambda) - \phi(P_{l_1-1,l_2,\cdots,l_t,0},\lambda).$$

Thus the result follows since both $\phi(P_{l_1,l_2,\cdots,l_t,0},\lambda)$ and $\phi(P_{l_1-1,l_2,\cdots,l_t,0},\lambda)$ contain the factor $(\lambda-2)^{t-1}$ by induction hypothesis. Similarly, by induction on l_{t+1}, we can show that 2 is an eigenvalue of $P_{l_1,l_2,\cdots,l_{t+1}}$ with multiplicity at least $t-1$. Thus the result follows.

Now we show that $\rho_s(P_{l_1,l_2,\cdots,l_k}) = 2$ and 2 is an eigenvalue of $P_{l_1,l_2,\cdots,l_k}$ with multiplicity exactly $k-2$ for $k \geq 3$. Assume to the contrary that $\rho_s(P_{l_1,l_2,\cdots,l_k}) > 2$ or 2 is an eigenvalue of $P_{l_1,l_2,\cdots,l_k}$ with multiplicity at least $k-1$. Then, in $P_{l_1,l_2,\cdots,l_k}$, there has at least $k-1$ eigenvalues no less than 2, that is, $\lambda_{k-1}(P_{l_1,l_2,\cdots,l_k}) \geq 2$. On the other hand, from Observation 1 we can delete $k-2$ outer vertices from $P_{l_1,l_2,\cdots,l_k}$ such that the resultant graph is of $P_{l_1',l_2'}$ with $l_1'+l_2' = n-k$. By Lemma 1,

$$\lambda_1(P_{l_1',l_2'}) \geq \lambda_{k-1}(P_{l_1,l_2,\cdots,l_k}) \geq 2,$$

which is a contradiction to Lemma 6. ■

By this lemma, we find that an oriented graph whose skew spectral radius is bounded by 2 may contain an arbitrary number of cycles, a much difference from that of undirected graphs.

3 The C_4-free oriented bicyclic graphs whose skew spectral radius does not exceed 2

In this section, we determine all the C_4-free oriented bicyclic graphs whose skew-spectral radius does not exceed 2.

Firstly, we introduce more notations. Denote by T_{l_1,l_2,l_3} the starlike tree with exactly one vertex v of degree 3, and $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, where P_{l_i} is the path of order $l_i (i = 1,2,3)$.

Due to Smith all undirected graphs whose (adjacency) spectral radius is bounded by 2 are completely determined as follows.

Lemma 8. ([3] or [8]) *All undirected graphs whose spectral radius does not exceed* 2 *are* $C_m, P_{0,n-4,0}$, $T_{2,2,2}$, $T_{1,3,3}$, $T_{1,2,5}$ *and their subgraphs, where* $m \geq 3$ *and* $n \geq 5$.

By Lemma 4, to study the spectrum properties of an oriented graph, we need only concern the parity of those cycles with length even. Moreover, Shader and So show that $S(G^\sigma)$ has the same spectrum as that of its underlying tree for any oriented tree G^σ; see Theorem 2.5 of [7]. Consequently, combining with Lemma 2 the skew-spectral radius of each oriented graph whose underlying graph is described as Lemma 8, regardless the orientation of the oriented cycle C_n^σ, does not exceed 2.

Therefore, we in the following focus on oriented graphs containing cycles. Let $C_m = v_1v_2\cdots v_mv_1$ be a cycle on m vertices and $P_1, P_2, \cdots, P_m$ be m paths with length $l_1, l_2, \cdots, l_m$, respectively. (Perhaps some of them are empty.) Denote by $C_m^{l_1,l_2,\cdots,l_m}$ the unicyclic undirected graph obtained from C_m by joining v_i to a pendent vertex of P_i for $i = 1, 2, \cdots, m$. For convenience, suppose without loss of generality that $l_1 = \max\{l_i : i = 1, 2, \cdots, m\}$, $l_2 \geq l_m$ and write $C_m^{l_1,l_2,\cdots,l_j}$ instead of the standard $C_m^{l_1,l_2,\cdots,l_j,0,\cdots,0}$ if $l_{j+1} = l_{j+2} = \cdots = l_m = 0$.

Denote by $\theta_{a,b,c}$ the undirected bicyclic graph obtained from paths P_a, P_b, P_c by identifying the three initial vertices and terminal vertices of them, where $\min\{a,b,c\} \geq 2$ and at most one of them is 2, and by $\infty_m^{p,q}(m \geq 1, p, q \geq 3)$ the undirected bicyclic graph obtained from cycles C_p and C_q joined by a path P_m.

Henceforth we will briefly write $\widehat{G}$ instead of the normal notation G^σ, an oriented graph, if no conflict exists there.

Let $\widehat{G}$ be an oriented graph with the property

$$\rho_s(\widehat{G}) \leq 2. \tag{3.1}$$

The property (3.1) is hereditary because, as a direct consequence of Lemma 1, for any induced subgraph $H \subset G$, $\widehat{H}$ also satisfies (3.1). The inheritance(hereditary) of property (3.1) implies that there are minimal connected graphs that do not obey (3.1); such graphs are called *forbidden subgraphs*.

On oriented unicyclic graphs, we have

Lemma 9. *Let* $\widehat{G}$ *be a* C_4*-free oriented unicyclic graph with* $\rho_s(\widehat{G}) \leq 2$. *Then* $\widehat{C}_3^3$, $\widehat{C}_3^{1,1}$, $\widehat{C}_3(2)$, $\widehat{C}_3^1(2)$, $\widehat{C}_5^1$, $\widehat{C}_7^1$ *are forbidden, where each induced even cycle is negative and* $\widehat{C}_3(2)$ *(or* $\widehat{C}_3^1(2)$*) denotes the oriented graph obtained by adding two pendent vertices to a vertex (or the pendent vertex)of* $\widehat{C}_3$ *(or* $\widehat{C}_3^1$*).*

For oriented bicyclic graphs, it is easy to verify that

Lemma 10. *Let* $\widehat{G}$ *be a* C_4*-free oriented bicyclic graph with* $\rho_s(\widehat{G}) \leq 2$. *Then* $\widehat{\infty}_1^{3,3}$, $\widehat{\infty}_2^{3,3}$, $\widehat{\theta}_{4,4,4}$ *are forbidden, where each induced even cycle is negative.*

Combining with Lemma 10 and the fact that $\rho_s(\widehat{G}) > 2$ if the oriented tree $\widehat{G}$ contains an arbitrary tree described as Lemma 8 as a proper subgraph, we have the following main result.

Theorem 1. *Let $\widehat{G}$ be a C_4-free oriented bicyclic graph with $\rho_s(\widehat{G}) \leq 2$. Then $\widehat{G}$ is $\widehat{\theta}_{3,5,5}$ or its induced oriented bicyclic subgraphs, where each induced even cycle is negative.*

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