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Direct Similarity Solution Method and Comparison with the Classical Lie Symmetry Solutions

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Abstract

We study the general applicability of the Clarkson–Kruskal's direct method, which is known to be related to symmetry reduction methods, for the similarity solutions of nonlinear evolution equations (NEEs). We give a theorem that will, when satisfied, immediately simplify the reduction procedure or ansatz before performing any explicit reduction expansions. We shall apply the method to both scalar and vector NEEs in either 1+1 or 2+1 dimensions, including in particular, a variable coefficient KdV equation and the 2+1 dimensional Khokhlov–Zabolotskaya equation. Explicit solutions that are beyond the classical Lie symmetry method are obtained, with comparison discussed in this connection.

1 Introduction

It is now well know that the classical Lie symmetry method can be used to find similarity solutions systematically, see, e.g., [1]. Recent years also saw a resurgence [2] of the so-called nonclassical Lie method due to Bluman and Cole [3]. Related to these two methods is also a direct similarity reduction method, introduced by Clarkson and Kruskal [4]. It is shown in various literature that the direct similarity reduction may lead to similarity solutions by classical and nonclassical Lie methods [1–12]. Some connections have also been established for the direct method and the nonclassical method [11,13].

Although the direct method has been successfully applied to a number of interesting nonlinear evolution equations (NEEs), the reduction procedure is highly equation-dependent. In this work, we shall introduce and prove a reduction theorem that will help us make some critical reduction ansatz *without* having to do any tedious calculation first. In other words, by just the inspection of a nonlinear PDE, the theorem may enable us to immediately make certain reductions which will significantly reduce the amount of calculation required to eventually find the similarity solutions. It is worth noting that even if the solutions are obtainable via the classical Lie method, the calculations in the direct method under the reduction theorem to be introduced in section 2 often turn out to be simpler.

This paper is organized as follows. In section 2, we shall first introduce several notations and definitions as the preliminaries and then set up our main reduction theorem there. The mKdV type and coupled KdV equations are then exemplified for direct applications. Then in section 3, we shall study the variable coefficient KdV equation and the 2+1 dimensional Khokhlov-Zabolotskaya (KZ) equation. New similarity solutions that are beyond the classical Lie method are obtained there. Finally, a brief summary is made in section 4.

2 Reduction Theorem

Let us consider the following system of nonlinear PDEs of a polynomial type

$$\mathbf{F}[\mathbf{u}] \equiv (F_1[\mathbf{u}], ..., F_m[\mathbf{u}]) = 0 \tag{2.1}$$

with $\mathbf{u}=(u_1, ..., u_m)$ and $\mathbf{x}=(x_1, ..., x_n)$. Let $\mathbf{N}$ be the set of non-negative integers and let $I_j^{(i)} = (I_{j,1}^{(i)}, ..., I_{j,n}^{(i)}) \in \mathbf{N}^n$. Then the *index* for a typical term

$$(\partial_{\mathbf{x}}^{I_1^{(1)}} u_1) \cdots (\partial_{\mathbf{x}}^{I_{K_1}^{(1)}} u_1) \cdots (\partial_{\mathbf{x}}^{I_1^{(m)}} u_m) \cdots (\partial_{\mathbf{x}}^{I_{K_m}^{(m)}} u_m) \tag{2.2}$$

with obviously $\partial_{\mathbf{x}}^{I_j^{(i)}} u_k \equiv \partial_{x_1}^{I_{j,1}^{(i)}} \cdots \partial_{x_n}^{I_{j,n}^{(i)}} u_k$ has the form

$$I =< I_1^{(1)}, ..., I_{K_1}^{(1)} \,|\, I_1^{(2)}, ..., I_{K_2}^{(2)} \,|\, \cdots \,|\, I_1^{(m)}, ..., I_{K_m}^{(m)} > . \tag{2.3}$$

We define the *order* and the *rank* of I, respectively, by

$$|I| = K_1 + \cdots + K_m \ , \quad \|I\| = \sum_{i,j,k} I_{j,k}^{(i)} \, . \tag{2.4}$$

Index I in (2.3) is termed *canonical* if it is 'lexically' ordered: one has for any i and j either $\|I_j^{(i)}\| = \|I_{j+1}^{(i)}\|$ and $I_{j,k}^{(i)} = I_{j+1,k}^{(i)}$ $(k = 1, .., L)$ with $I_{j,L+1}^{(i)} < I_{j+1,L+1}^{(i)}$ for some L or simply $\|I_j^{(i)}\| < \|I_{j+1}^{(i)}\|$. Because every term of the form (2.2) will have a unique canonical index, we shall from now on assume that all indices are canonical.

For any index I of form (2.3), we may rewrite it as

$$I =< I^{(1)} \,|\, I^{(2)} \,|\, \cdots \,|\, I^{(m)} >= I_{(1)} \oplus I_{(2)} \oplus \cdots \oplus I_{(n)} \tag{2.5}$$

where $I^{(i)} =< I_1^{(i)}, .., I_{K_i}^{(i)} >$ corresponds to u_i alone and $I_{(j)}$ is obtained from I by setting all the non-x_j indices to 0. For example, the term $u_1 u_{1,x_1} u_{1,x_1 x_2} u_{3,x_2,x_2}$ for $m = 3$ and $n = 2$ has the corresponding index $I =< (0, 0), (1, 0), (1, 1)| \,|(0, 2) >$ with

$$I^{(1)} =< (0, 0), (1, 0), (1, 1) >, \quad I^{(2)} =<>, \quad I^{(3)} =< (0, 2) >,$$

$$I_{(1)} =< (0, 0), (1, 0), (1, 0)| \,|(0, 0) >, \quad I_{(2)} =< (0, 0), (0, 0), (0, 1)| \,|(0, 2) > \ .$$

For convenience, we shall denote the terms corresponding to I and $I^{(i)}$ by $\partial_{\mathbf{x}}^{I} \circ \mathbf{u}$ and $\partial_{\mathbf{x}}^{I^{(i)}} \circ u_i$, respectively. When no confusion is to occur, we may also denote the index I in (2.3) simply by its *totality form*

$$I =< \|I_1^{(1)}\|, ..., \|I_{K_1}^{(1)}\| \,|\, ... \,|\, \|I_1^{(m)}\|, ..., \|I_{K_m}^{(m)}\| > \tag{2.6}$$

and may even use an index and the corresponding term interchangeably. Thus, we may sometimes just write, for instance, $\|u_{1,x_1}u_{3,x_2x_2}\|=3$.

Let $\mathbf{I}=\bigcup_{i=1}^{m}\mathbf{I}^{(i)}$ and $\mathbf{I}^{(i)}$ be the set of indices corresponding to all the terms in $F_i[\mathbf{u}]$. An index $I\in\mathbf{I}$ is called *pure* if there are no mixed (i.e., cross) partial derivatives in any factors in $\partial_{\mathbf{x}}^{I}\circ\mathbf{u}$. An index I is said to be *reachable* from another index J w.r.t. $\mathbf{I}$, if $I,\ J\in\mathbf{I}^{(i_0)}$ for some i_0 and that I can be obtained from J, at least in terms of the totality form, by decreasing some subindices in J and perhaps removing some entries $I_j^{(i)}$ completely. An index I is regarded as being *prime* w.t.r. $\mathbf{I}$ if I is not reachable from any $J\in\mathbf{I}\backslash\{I\}$. Index $\tilde{I}$ is said to be a *pure tail* of I if $\tilde{I}$ can be obtained by decreasing just one positive subindex $I_{j,k}^{(i)}$ by 1 and $I_j^{(i)}$ is a pure index. Index I^* will be called a *shrunk index* of I if I^* is obtained by removing from I exactly one entry, say $I_j^{(i)}$. An index I is said to have a *maximal rank* in $\mathbf{I}$ if $\|J\|\le\|I\|$ whenever $I,J\in\mathbf{I}^{(i)}$. And finally, an index I is said to be a *sole index of maximal rank* in $\mathbf{I}$ if $\|J\|<\|I\|$ whenever $I,J\in\mathbf{I}^{(i)}$, $I\ne J$, and J is reachable from I. With the above definitions, we are ready to introduce the following

Reduction Theorem Suppose a system of nonlinear PDEs of a polynomial type is given by (2.1) and $\mathbf{I}=\bigcup_{i=1}^{m}\mathbf{I}^{(i)}$ is the corresponding index set with $\mathbf{I}^{(i)}$ containing indices of all terms in $F_i[\mathbf{u}]$. We seek reduction of the PDEs into ODEs w.r.t. $z(\mathbf{x})$ with $\partial_{x_i}z(\mathbf{x})\ne 0$.

(i) Suppose that

$$u_i(\mathbf{x})=U_i(\mathbf{x},w_i(z(\mathbf{x}))),\qquad i=1,...,m, \tag{T_1}$$

reduce the PDEs into ODEs in $w_i(z)$'s. Suppose for any i, $1\le i\le m$, there exists at least one sole index of maximal rank such that $\|I^{(i)}\|>0$, then all similarity solutions of the PDEs induced by (T_1) can also be obtained by the reduction

$$u_i(\mathbf{x})=\alpha_i(\mathbf{x})+\beta_i(\mathbf{x})w_i(z(\mathbf{x})),\qquad i=1,...,m\,. \tag{T_2}$$

(ii) Suppose $I,J\in\mathbf{I}^{(i_0)}$ for some i_0 are two prime indices, $|I|\ne|J|$, and the coefficients of the corresponding terms in $F_{i_0}[\mathbf{u}]$ are constant multiple of each other. Then (T_2) reduces the PDEs into ODEs that implies we may set without loss of generality

$$\beta_1^{|J^{(1)}|-|I^{(1)}|}\cdots\beta_m^{|J^{(m)}|-|I^{(m)}|}=z_{x_1}^{\|I_{(1)}\|-\|J_{(1)}\|}\cdots z_{x_n}^{\|I_{(n)}\|-\|J_{(n)}\|} \tag{T_3}$$

where $I^{(i)}$, $J^{(i)}$, $I_{(i)}$ and $J_{(i)}$ are defined via (2.5). In the case of m=1, (T_3) reads

$$\beta(\mathbf{x})=z_{x_1}^{\frac{\|I_{(1)}\|-\|J_{(1)}\|}{|J|-|I|}}\cdots z_{x_n}^{\frac{\|I_{(n)}\|-\|J_{(n)}\|}{|J|-|I|}}\,. \tag{T_4}$$

(iii) Suppose there exist i_0, j_0, and $I\in\mathbf{I}^{(k_0)}$ such that

$$\beta_{j_0}(\mathbf{x})=\gamma(x_1,..,x_{i_0-1},x_{i_0+1},..,x_n)z_{x_{i_0}}^{p}(\mathbf{x})\,,\quad p\ge 0\,, \tag{T_5}$$

index I has a pure tail $\tilde{I}$ w.r.t. u_{j_0} and x_{i_0} and that $\tilde{I}$ is not reachable elsewhere in $\mathbf{I}$. Then (T_2) reduces the PDEs into ODEs that implies

$$z(\mathbf{x})=\theta(x_1,..,x_{i_0-1},x_{i_0+1},..,x_n)x_{i_0}+\sigma(x_1,..,x_{i_0-1},x_{i_0+1},..,x_n)\,. \tag{T_6}$$

(iv) Suppose there exists an $I \in \mathbf{I}$ such that it has a shrunk index I^* (w.r.t. u_{i_0}) of same rank (i.e. $\|I^*\| = \|I\|$ and $\partial_{\mathbf{x}}^I \circ \mathbf{u} = u_{i_0}\partial_{\mathbf{x}}^{I^*} \circ \mathbf{u}$) that is not reachable elsewhere. Then (T_2) reduces the PDEs into ODEs that implies we may set without loss of generality

$$\alpha_{i_0}(\mathbf{x}) = 0 \ . \qquad (T_7)$$

The complete proof is somewhat long, we shall thus give only the proof of (ii) whose idea will be used later on as well. Notice that $I, J \in \mathbf{I}^{(i_0)}$ are prime implies

$$\begin{aligned} F_{i_0}[\mathbf{u}] = \quad & f(\mathbf{x})\partial_{\mathbf{x}}^I \circ \mathbf{u} + g(\mathbf{x})\partial_{\mathbf{x}}^J \circ \mathbf{u} + \sum_{S\in\mathbf{I}^{(i_0)}\backslash\{I,J\}} B_S(\mathbf{x})\partial_{\mathbf{x}}^S \circ \mathbf{u} = \\ & f(\mathbf{x})\beta_1^{|I^{(1)}|}\cdots\beta_m^{|I^{(m)}|} z_{x_1}^{\|I_{(1)}\|}\cdots z_{x_n}^{\|I_{(n)}\|}\mathbf{w}^A + \\ & g(\mathbf{x})\beta_1^{|J^{(1)}|}\cdots\beta_m^{|J^{(m)}|} z_{x_1}^{\|J_{(1)}\|}\cdots z_{x_n}^{\|J_{(n)}\|}\mathbf{w}^B + \dots , \end{aligned}$$

where $\mathbf{w}^A$ and $\mathbf{w}^B$ are certain power products of w_k's and their derivatives, which are independent of each other and of all other terms on the right hand-side of the above equation. Hence in order that the above equation be an ODE, we need

$$g(\mathbf{x})\beta_1^{|J^{(1)}|}\cdots\beta_m^{|J^{(m)}|} z_{x_1}^{\|J_{(1)}\|}\cdots z_{x_n}^{\|J_{(n)}\|} = f(\mathbf{x})\beta_1^{|I^{(1)}|}\cdots\beta_m^{|I^{(m)}|} z_{x_1}^{\|I_{(1)}\|}\cdots z_{x_n}^{\|I_{(n)}\|}\Gamma(z)$$

for some function $\Gamma(z)$. Thus if $f(\mathbf{x})/g(\mathbf{x})$=constant and $|J| \neq |I|$, then we can assume without loss of generality that $|J^{(j_0)}| \neq |I^{(j_0)}|$. By absorbing $\Gamma(z)f(\mathbf{x})/g(\mathbf{x})$ into $w_{j_0}(z)$, we finally obtain (T_3). This completes the proof of (ii).

In order to exemplify the definitions and results related to the Reduction Theorem, we first briefly consider a simple NEE of the mKdV type

$$F[u] \equiv u_t + u^s u_x + u_{xxx} = 0, \qquad s \geq 2 \ . \qquad (2.7)$$

The index set is obviously $\mathbf{I}=\{<(0,1)>, <\overbrace{(0,0),...,(0,0)}^{s},(1,0)>, <(0,3)>\}$ if we take $\mathbf{x}$=(x,t). In fact, we shall in this case write it simply in the *totality form* $\mathbf{I}=\{<1>, <0,...,0,1>, <3>\}$. We now observe that 1° Index $<0,..,0,1>$ or term $u^s u_x$ has the *order* $p+1$ and the *rank* 1. 2° All indices in $\mathbf{I}$ for (2.7) are *pure indices*. 3° Index $<1>$ is *reachable* from either $<0,..,0,1>$ or $<3>$, which is tantamount to saying u_t is reachable from either $u^s u_x$ or u_{xxx}. 4° Indices $<0,..,0,1>$ and $<3>$ are both *prime* w.r.t. $\mathbf{I}$. 5° Index $<3>$ has a *pure tail* $<2>$ not reachable elsewhere, i.e., u_{xx} is a pure tail of u_{xxx} and u_{xx} is not reachable from u_t or $u^s u_x$. 6° $<\overbrace{0,..,0}^{s-1},1>$ is a *shrunk index* of $<\overbrace{0,..,0}^{s},1>$ and is unreachable elsewhere due to $s \geq 2$. 7° Index $<3>$ or term u_{xxx} has the *maximal rank* 3 and is also the *sole* index or term of maximal rank. Hence, from the above theorem, a reduction of (2.7) into ODE by $u = U(x,t,w(z(x,t))$ can be simplified without loss of generality to $u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t))$; 4° implies

$$\beta(x,t) = z_x^{\frac{\|<3>\|-\|<0,..,0,1>\|}{|<0,..,0,1>|-|<3>|}} = z_x^{2/s} \ ;$$

5° implies $z(x,t) = \theta(t)x + \sigma(t)$ and 6° implies $\alpha(x,t) = 0$. Hence, we conclude that

$$u(x,t) = \theta(t)^{2/s} w(\theta(t)x + \sigma(t)) \tag{2.8}$$

is the most general form to reduce (2.7) to an ODE with the direct method. Inserting (2.8) into (2.7), one obtains with *simple* calculations that (2.8) is the solution of (2.7) if

$$\theta(t) = \frac{A_0^s}{(t+t_0)^{1/3}}, \qquad \sigma(t) = \frac{B_0}{(t+t_0)^{1/3}} - C_0$$

and that $w(z)$ is a solution of

$$w''' + w^s w' - \frac{2}{3sA_0^{3s}} w - \frac{1}{3A_0^{3s}}(z + C_0)w' = 0 \, .$$

In comparison with the original CK procedure [4] for s=2, our Reduction Theorem will enable us to start with a form similar to (2.8) *without* any messy expansion first. Thus, the application of the theorem significantly reduces the overall bulk of tedious calculations. We note that the above derived similarity solution via (2.8) can also be obtained by the classical Lie method. However, our procedure in this case seems much simpler.

Let us now examine a simple *system* of NEEs, the coupled-KdV equations,

$$ru_t + u_{xxx} + 6uu_x + 2vv_x = 0, \quad v_t + v_{xxx} + 3uv_x = 0 \tag{2.9}$$

which is equivalent to the form given in [14] under some rescaling. In this case, we observe that 1° u_{xxx} and v_{xxx} are both sole terms of maximal rank. Hence the direct method with the form (T_1) implies one may just set u=α_1+$\beta_1 U(z)$ and v=α_2+$\beta_2 V(z)$. 2° uu_x, u_{xxx} and vv_x are all prime in (2.9a). From (T_3), the first pair implies $\beta_1 = z_x^2$ and the second pair implies $\beta_2 = z_x^2$. 3° u_{xxx} has pure tail u_{xx} unreachable elsewhere implies $z = \theta(t)x + \sigma(t)$. 4° vv_x has shrunk form v_x unreachable elsewhere implies we may set $\alpha_2 \equiv 0$. To summarize, we may set

$$u = a(x,t) + \theta(t)^2 U(z), \quad v = \theta(t)^2 V(z), \quad z = \theta(t)x + \sigma(t) \, . \tag{2.10}$$

For simplicity, we shall denote by $f \sim g$ for any f and g if $f = g\Gamma(z)$ for some $\Gamma(z)$. Eq.(2.9) be an ODE when inserted by (2.10) requires $\dot{\theta} \sim \theta^4$ and $3a\theta + \dot{\theta}x + \dot{\sigma} \sim \theta^3$, whose solutions, when inserted back into (2.9), will give rise to an extra equation $\theta x - 3\dot{\sigma}(t+t_0) \sim 1$. The solutions of these three equations are

$$\theta(t) = A_0/(t+t_0)^{1/3}, \quad \sigma(t) = B_0/(t+t_0)^{1/3} - C_0, \quad a(x,t) = -(\dot{\theta}x + \dot{\sigma})/3\theta \tag{2.11}$$

which will reduce (2.9) into two ODEs

$$U''' + 2VV' + 6UU' + \frac{(2-r)(z+C_0)}{3A_0^3} U' + \frac{2(1-r)}{3A_0^3} U + \frac{(2-3r)}{27A_0^6}(z + C_0) = 0 \, ,$$

$$V''' + 3UV' - \frac{2V}{3A_0^3} = 0 \, . \tag{2.12}$$

We note that we can solve U from (2.12b) and insert it into (2.12a) to derive a single ODE, whose solution will give rise to similarity solutions to the coupled KdV equations (2.9) via (2.10) and (2.11).

We note that the Reduction Theorem is directly applicable to various NEEs, among them are the Sawada–Kotera, KdV, Burgers, Boussinesq, Kupershmidt, and KP, and many other equations. For some NEEs such as the Fitzhugh–Nagumo equation [9], the theorem is only partially applicable. All these mentioned (apart from the Sawada–Kotera) equations are already solved without the use of the above theorem [4–12]. However, the use of the Reduction Theorem will significantly reduce the amount of entailed calculation.

3 Variable coefficient KdV and Khokhlov–Zabolotskaya equations

We now concentrate on similarity solutions of the following variable coefficient KdV equation [15]

$$u_t + t^n u u_x + t^m u_{xxx} = 0, \qquad m, n \in \mathbf{R}\,. \tag{3.1}$$

Eq.(3.1) reduces to the KdV equation in the trivial case of m=n=0 and also reduces via $u = \sqrt{t}v$ for the case of $m = 0$ and $n = -1/2$ to the Cylindrical KdV equation $v_t + vv_x + v_{xxx} + v/(2t) = 0$. The similarity solutions via classical Lie symmetries are already obtained in [15]. Our purpose here is to derive new solutions with the direct method, with the assistance of the Reduction Theorem.

In applying the direct method to (3.1), we can easily see that only (i) and (iii) of the Reduction Theorem are directly applicable, though (ii) is only indirectly applicable. Nevertheless, we shall in this section work through the direct method to obtain two new cases of similarity solutions:

(I) If $m \neq -1$, the (3.1) has the similarity solution

$$u(x,t) = \alpha_0 + A_0^2 (t^{m+1} + t_0)^{\frac{m-3n-2}{3(m+1)}} w(z(x,t)) \tag{3.2}$$

where $t_0 \equiv 0$ if $n \neq m$,

$$z(x,t) = \frac{A_0}{(t^{m+1}+t_0)^{1/3}} \left\{ x + \int^t \Big[\frac{A_0^2 B_0 \tau^m}{(\tau^{m+1}+t_0)^{2/3}} - \alpha_0 \tau^n \Big] d\tau + C_0 \right\}, \tag{3.3}$$

$w(z)$ is a solution of

$$w''' + ww' + (B_0 - \frac{m+1}{3A_0^3} z) w' + (\frac{m-3n-2}{3A_0^3}) w = 0 \tag{3.4}$$

and that α_0, t_0, A_0, B_0 and C_0 are all real constants.

(II) If $m = n = -1$, then (3.1) has the similarity solution

$$u(x,t) = \alpha_0 + A_0^2 (\ln t + B_0)^{-2/3} w(z) \tag{3.5}$$

where

$$z(x,t) = \frac{A_0}{(\ln t + B_0)^{1/3}} \left\{ x + \int^t \Big[\frac{A_0^2 D_0}{(\ln \tau + B_0)^{2/3}} - \alpha_0 \Big] \frac{1}{\tau} d\tau + C_0 \right\}, \tag{3.6}$$

$w(z)$ is a solution of

$$w''' + ww' + (D_0 - \frac{z}{3A_0^3})w' - \frac{2}{3A_0^3}w = 0 \tag{3.7}$$

and that α_0, t_0, A_0, B_0, C_0 and D_0 are all real constants.

Both (I.) and (II.) are essentially new solutions, although by setting in (I.) $\alpha_0 = B_0 = C_0 = t_0 = 0$, one obtains a known self-similarity solution not obtainable from the classical Lie method according to [15].

We now proceed with the derivation of (3.2)–(3.7). As usual, we start with $u = U(x, t, w(z(x, t)))$. Because $< 3 >$ is a sole index of maximal rank for (3.1) and that $< 3 >$ has the pure tail $< 2 >$ unreachable elsewhere, we may, according to the Reduction Theorem in section 2, assume simply

$$u = \alpha(x,t) + \beta(x,t)w(z(x,t)), \qquad z(x,t) = \theta(t)x + \sigma(t) \; . \tag{3.8}$$

We thus obtain from (3.8) and (3.1)

$$\begin{aligned} u_t + \quad & t^n uu_x + t^m u_{xxx} = t^m \beta\theta^3 w''' + 3t^m \beta_x \theta^2 w'' + t^n \beta^2 \theta ww' + t^n \beta\beta_x w^2 + \\ & \Big[\alpha\beta\theta t^n + 3\beta_{xx}\theta t^m + \tfrac{\beta}{\theta}(\dot\theta z + \theta\dot\sigma - \dot\theta\sigma)\Big] w' + [(\alpha_x\beta + \alpha\beta_x)t^n + \beta_{xxx}t^m + \\ & \beta_t] w + (\alpha\alpha_x t^n + \alpha_{xxx}t^m + \alpha_t) = 0 \; . \end{aligned} \tag{3.9}$$

Similar to the proof of (ii) of the reduction theorem, by comparing the coefficients of w''' and ww', we obtain $t^n\beta^2\theta \sim t^m\beta\theta^3$. Hence, we may set

$$\beta(x,t) = t^{m-n}\theta(t)^2 \; . \tag{3.10}$$

In order that (3.9) be an ODE in $w(z)$, we require

$$\begin{aligned} t^n\alpha\beta\theta + (\dot\theta z + \theta\dot\sigma - \dot\theta\sigma)\beta/\theta \quad &\sim \quad t^m\beta\theta^3, \\ t^n\alpha_x\beta + \beta_t \quad &\sim \quad t^m\beta\theta^3, \\ t^n\alpha\alpha_x + t^m\alpha_{xxx} + \alpha_t \quad &\sim \quad t^m\beta\theta^3 \; . \end{aligned} \tag{3.11}$$

First we notice that $\alpha_x = \alpha_t = 0$ solves (3.11c). Hence we set $\alpha(x,t) = \alpha_0$ and insert it along with (3.10) into (3.11a). We then obtain

$$t^n\alpha\beta\theta + (\theta\dot\sigma - \dot\theta\sigma)\beta/\theta - z\beta\dot\theta/\theta = t^m\beta\theta^3(Az + B)$$

for some constant A and B. In other words, we need to solve

$$\frac{d}{dt}\theta = At^m\theta^4, \qquad t^n\alpha + \frac{d}{dt}\Big(\frac{\sigma}{\theta}\Big) = t^m\theta^2 B \; . \tag{3.12}$$

The solution of (3.12) will be separated into two cases. In the first case, $m \neq -1$, we obtain from (3.12a) and (3.11b)

$$\theta(t) = \frac{A_0}{(t^{m+1} + t_0)^{1/3}}, \qquad \beta(x,t) = \beta_0(t^{m+1} + t_0)^p \tag{3.13}$$

for some constant p, t_0, A_0 and β_0. The compatibility of (3.13) with (3.10) then requires

$$p = \frac{m-3n+2}{3(m+1)}, \quad \beta_0 = A_0^2, \quad t_0 = 0 \quad \text{if } m \neq n\ . \tag{3.14}$$

Inserting (3.8) into (3.1) with (3.13), (3.14) and (3.12b), i.e., inserting (3.2) into (3.1) with (3.3), we obtain simply (3.4). Hence (I.) is a similarity solution of (3.1).

In the second case, $m = -1$, the solution of (3.12a) and (3.11b) reads

$$\theta(t) = \frac{A_0}{(\ln t + B_0)^{1/3}}, \qquad \beta(x,t) = \beta_0(\ln t + B_0)^p \tag{3.15}$$

whose compatibility with (3.10) gives

$$n = m = -1, \quad p = -\frac{2}{3}, \quad \beta_0 = A_0^2\ . \tag{3.16}$$

Inserting (3.8) into (3.1) with (3.15), (3.16) and (3.12b), i.e., inserting (3.5) into (3.1) with (3.6), we obtain (3.7). Hence (II.) is also a similarity solution of (3.1). We have thus obtained two new similarity solutions for (3.1). As a result, we have also obtained a new similarity solution for the Cylindrical KdV equation. We note that there is another reduction procedure which does not need to *impose* any conditions like $\alpha \equiv \alpha_0$. Although the final results are equivalent, the procedure explicitly presented above turns out to be simpler. Hence we see that an early reduction may not be the best choice in some cases in the sense of solution convenience.

We now move to study very briefly a 2+1 dimensional NEE, the KZ equation [16]

$$u_{xt} + (uu_x)_x + u_{yy} = 0\ . \tag{3.17}$$

Notice that there are four indices of maximal rank and that although $I = < 0, 2 >$ and $J = < 1, 1 >$ are prime, they are not very useful due to $\|I\| = \|J\|$ and $|I| = |J|$. Hence, eq.(3.17) will be an example for which the Reduction Theorem is not directly applicable. For the direct method, we shall nevertheless still seek the solution in the form of $u = \alpha(x,y,t) + \beta(x,y,t)w(z(x,y,t))$. Since the reduction procedure for (3.17) to become an ODE is somewhat technical and lengthy, we shall instead report here only a special case

$$u = a(t)y + b(t) + \frac{c(t)}{f(t)} w(f(t)x + g(t)y^2 + h(t)y + m(t)). \tag{3.18}$$

Since (3.18) reduces (3.17) into

$$\begin{aligned}\Big[\ u_{xt} + (uu_x)_x + u_{yy}\Big]/c(t)^2 &= ww'' + (w')^2 + [(\dot{c}f + 2cg)/(fc^2)]w' + \\ &\Big\{[(f\dot{g} - \dot{f}g + 4g^2)/(cf)]y^2 + [(af^2 + f\dot{h} - \dot{f}h + 4gh)/(cf)]y + \\ &[\dot{f}(fx + gy^2 + hy + m)/(cf) + (bf^2 + f\dot{m} - \dot{f}m + h^2)/(cf)]\Big\} w'' = 0\ ,\end{aligned} \tag{3.19}$$

eq.(3.19) be an ODE in w requires

$$\dot{f} = Acf, \quad 2cg + \dot{c}f = Bc^2 f, \quad f\dot{g} - \dot{f}g + 4g^2 = 0,$$

$$f^2a + f\dot{h} - \dot{f}h + 4gh = 0, \quad f^2b + f\dot{m} - \dot{f}m + h^2 = Ccf \tag{3.20}$$

for some constant A, B and C. Notice that eqs.(3.20a,b,c) give for $f(t)$ the following equation

$$3A^2f^2(f'')^2 - 3A(A+B)f(f')^2f'' - A^2f^2f'f''' + (A+2B)(A+B)(f')^4 = 0\ . \tag{3.21}$$

It is easy to see that for any solution $f(t)$ of (3.21), functions $c(t)$, $g(t)$, $a(t)$ and $b(t)$ can be obtained immediately from (3.20a,b,d,e), respectively, with $m(t)$ being arbitrary. The KZ equation (3.17) will then reduce to

$$(w + Az + C)w' + (B - A)w = D$$

for some constant D. We note that a particular solution of (3.21) is $f(t) = (t + t_0)^s$ with $(A + sB)(A + 2sB) = 0$, which will correspond to a class of similarity solutions for (3.17). The simplest nontrivial case is given by $s = -1$ and $A = B = -1$ and $a = b = 0$, for which we have $c(t) = f(t) = 1/(t+t_0)$, $g(t) = 0$, $h(t) = y_0/(t+t_0)$ and $m(t) = z_0/(t+t_0)$. Hence we conclude that

$$u(x, y, t) = w\Big(\frac{x + y_0y + z_0}{t + t_0}\Big), \qquad (w - z + y_0^2)\frac{dw}{dz} = D, \tag{3.22}$$

for constant y_0, z_0, t_0 and D, is a similarity solution of (3.17). Incidentally, the solution of (3.22b) is determined by $D(w-(z+D)+y_0^2)\exp(w/D) = E$ (with constant E) for $D \neq 0$, and by $w(z) = z - y_0^2$ for $D = 0$. Obviously, these solutions are beyond those obtained by the classical Lie method in [15]. New nonclassical ansatzes for the Khohklov–Zabolotskaya equation were constructed in [17].

Conclusion

In this work, we re-examined the direct method of Clarkson–Kruskal. We presented a reduction theorem for the direct method so that a significant part of tedious calculations entailed by the direct method may be avoided for many NEEs. Direct applications were performed for the mKdV-type NEE and couple KdV equations, which give rise to similarity solution derivable from the classical Lie method. The variable coefficient KdV equation, transformable to the Cylindrical KdV equation in a special case, has been studied thoroughly with the direct method, for which the reduction theorem is only partially applicable. We have finally presented a special case of similarity solutions for the Khokhlov–Zabolotskaya equation, for which the reduction theorem is no longer directly applicable. The similarity solutions obtained in this work for the variable coefficient KdV equation and for the KZ equation seem new and not obtainable from the classical Lie symmetry method (see also [17]).

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