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Symmetry Reduction for a System of Nonlinear Evolution Equations

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Abstract

In this paper we obtain the maximal Lie symmetry algebra of a system of PDEs. We reduce this system to a system of ODEs, using some rank three subalgebras of the finite-dimensional part of the symmetry algebra. The corresponding invariant solutions of the PDEs are obtained.

1 Introduction

Systems of evolution equations describe various processes in physics, chemistry and biology [1, 2]. In the papers [3, 4] classes of systems of evolution equations are selected which are invariant with respect to some generalizations of the classical Galilei algebra. We consider the system proposed by W. Fushchych

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{m}\nabla u \nabla v + \dfrac{1}{2m} u \triangle v = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{2m}(\nabla v)^2 - \dfrac{\hbar^2}{2m}\dfrac{\triangle u}{u} = 0, \end{cases} \tag{1}$$

in four-dimensional time-space $\mathbf{R}(1,3)$, where $\hbar$ is the Planck's constant, $m \in \mathbf{R}$, $m \neq 0$. System (1) is obtained from the Schrödinger equation

$$i\psi_t = -\frac{\hbar}{2m}\triangle\psi$$

by the substitution $\psi = ue^{iv/\hbar}$.

From Theorem 1 it follows that the maximal symmetry algebra of system (1) contains the algebra $L = AG_3(3) \oplus < Z >$, where $AG_3(3)$ is the special Galilei algebra. Using the classification of subalgebras of the algebra $AG_3(3)$, carried out in [5], we obtain all (up to $Ad\,L$-conjugacy) I-maximal rank three subalgebras of the algebra L. For symmetry reduction of system (1) we use only subalgebras, whose projections onto $ASL(2,R)$ belong to $< T >$. In some cases considered the reduced system can be integrated, so that all corresponding solutions of system (1) can be constructed.

2 Maximal symmetry algebra of system (1)

We use the following notations

$$\delta_t = \frac{\delta}{\delta t}, \quad \delta_a = \frac{\delta}{\delta x_a}, \quad \delta_u = \frac{\delta}{\delta u}, \quad \delta_v = \frac{\delta}{\delta v} \quad (a = 1, 2, 3).$$

Theorem 1 *The maximal Lie symmetry algebra of system (1) is generated by the vector fields*

$$\begin{aligned} &S = t^2\partial_t + tx_a\partial_a - \frac{3}{2}tu\partial_u + \frac{m}{2}|\vec{x}|^2\partial_v, \\ &D = 2t\partial_t + x_a\partial_a - \frac{3}{2}u\partial_u, \quad T = \partial_t, \quad P_a = -\partial_a, \\ &G_a = t\partial_a + mx_a\partial_v, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad M = m\partial_v, \\ &Z = u\partial_u, \quad (a < b; \quad a, b = \overline{1, 3}) \end{aligned} \tag{2}$$

and the infinite-dimensional algebra

$$X = \rho\left(\cos\frac{\theta - v}{\hbar}\delta_u + \frac{\hbar}{u}\sin\frac{\theta - v}{\hbar}\delta_v\right), \tag{3}$$

where pair of functions $u = \rho(t, \vec{x})$, ($\rho(t, \vec{x}) \geq 0$) and $v = \theta(t, \vec{x})$ is an arbitrary solutions of system (1) (summation over repeated indices is assumed with a=1,2,3).

By direct calculations it is easy to verify that the operators M, P_a, G_a, J_{ab} $(a, b = 1, 2, 3)$, D, S, T generate the special Galilei algebra $AG_3(3)$ of the three-dimensional space. The operator Z commutes with each element of the algebra $AG_3(3)$. Let $U =< M, P_1, P_2, P_3, G_1, G_2, G_3 >$. Then

$$AG_3(3) = U \mathbin{\supset\!\!\!\!+} (AO(3) \oplus ASL(2, \mathbf{R})).$$

The algebra $AG_3(3)$ contains the Galilei algebras $AG_j(3)$ $(j = 0, 1, 2)$, where

$$AG_0(3) = U \mathbin{\supset\!\!\!\!+} AO(3), \qquad AG_1(3) = U \mathbin{\supset\!\!\!\!+} (AO(3) \oplus < T >),$$

$$AG_2(3) = U \mathbin{\supset\!\!\!\!+} (AO(3) \oplus < D, T >).$$

The symmetries can be used to build ansatzes which then reduce the equations of (1) to partial differential equations with fewer independent variables or even to ordinary differential equations. These ansatzes and reductions are based on subalgebra analysis of a finite-dimensional part of the symmetry algebra.

3 Classification of I-maximal subalgebras

The concept of I-maximal subalgebra was introduced in the paper [6].

Theorem 2 *The I-maximal rank three subalgebras of the algebra $L = AG_3(3) \oplus < Z >$, which have zero intersection with $< M, Z >$, are (up to the $Ad\,L$-conjugacy):*

(i) Subalgebras of the algebra $AG_0(3) \oplus < Z >$:

$$F_0 =< P_1, P_2, P_3 > \mathbin{\supset\!\!\!\!+} AO(3);$$

$F_1 =< G_1 + \alpha Z, P_2, P_3, J_{23} >,$ *where* $\alpha = 0, 1;$

$F_2 =< G_1 + P_1 + \alpha Z, G_2 + \beta Z, P_3 + \gamma Z > \quad (\alpha \geq 0, \beta \geq 0);$

$F_3 =< G_1 + \alpha Z, P_2 + Z, P_3 >; \qquad (\alpha \geq 0)$

$F_4 =< P_1 + Z, P_2, P_3, J_{23} > .$

(ii) Subalgebras of the algebra $AG_1(3) \oplus < Z >$ *with a nonzero projection onto* $< T >$*:*

$F_5 =< P_2, P_3, T + \frac{\alpha}{m} M + \beta Z, J_{23} >,$

where $\alpha = \pm m$ *or* $\alpha = 0$ *and* $\beta = 0, \pm 1;$

$F_6 =< P_2, P_3, T + G_1 + \alpha Z, J_{23} >;$

$F_7 =< P_3 + \alpha Z, J_{12} + \frac{\beta}{m} M + \gamma Z, T + \frac{\delta}{m} Z + \lambda Z >;$

$F_8 = AO(3) \oplus < T + \frac{\alpha}{m} M + \beta Z >;$

$F_9 =< T + G_1 + \alpha Z, P_2 + \beta Z, P_3 > \quad (\beta > 0);$

$F_{10} =< T + \frac{\alpha}{m} M + \beta Z, P_2 + Z, P_3 > .$

(iii) Subalgebras of the algebra $AG_2(3) \oplus < Z >$ *with a nonzero projection onto* $< D >$*:*

$F_{11} =< G_1, P_2, D + \alpha M + \beta Z >; \qquad F_{12} =< P_3, D + \alpha M + \beta Z, T >;$

$F_{13} =< P_2, P_3, J_{23}, D + \alpha M + \beta Z >;$

$F_{14} =< J_{12} + \alpha M + \beta Z, D + \gamma M + \delta Z, T >;$

$F_{15} =< P_3, J_{12} + \alpha D + \beta M + \gamma Z, T > \quad (\alpha > 0);$

$F_{16} =< P_3, J_{12} + \alpha M + \beta Z, D + \gamma M + \delta Z > \quad (\alpha \geq 0);$

$F_{17} = AO(3) \oplus < D + \alpha M + \beta Z > .$

(iv) Subalgebras of the algebra $AG_3(3) \oplus < Z >$*, whose projections onto* $ASL(2, \mathbf{R})$ *coincide with* $< S + T >$*:*

$F_{18} = AO(3) \oplus < S + T + \alpha M + \beta Z >;$

$F_{19} =< S + T + 2J_{12} + \alpha M + \beta Z, \; G_1 + P_2 + \sqrt{2} P_3, G_2 - P_1 - \sqrt{2} G_3 > .$

4 Reduction of system (1) by subalgebras of the algebra $\mathbf{AG_1(3)} \oplus < \mathbf{Z} >$. Exact solutions

For each of the subalgebras F_j ($j = \overline{1, 10}$) we give the corresponding ansatz and the reduced system. In some cases we also point out solutions of system (1), which are invariant under F_j.

4.1. $F_1: \ u = \exp\left(\frac{\alpha x_1}{t}\right)\varphi(\omega), \qquad v = \frac{m x_1^2}{2t} + \psi(\omega), \qquad \omega = t,$

$$2\omega\dot{\varphi} + \varphi = 0, \qquad \dot{\psi} - \frac{\hbar^2\alpha^2}{2m\omega^2} = 0.$$

The corresponding invariant solution of system (1) is of the form

$$u = \frac{C_1}{\sqrt{t}}\exp\left(\frac{\alpha x_1}{t}\right), \qquad v = \frac{m x_1^2}{2t} - \frac{\hbar^2\alpha^2}{2mt} + C_2.$$

4.2. $F_2: \ u = \exp\left(\frac{\alpha}{t-1}x_1 + \beta\frac{x_2}{t} - \gamma x_3\right)\varphi(\omega),$

$$v = \frac{m}{2(t-1)}x_1^2 + \frac{m}{2t}x_2^2 + \psi(\omega), \qquad \omega = t,$$

$$\begin{cases} 2\omega(\omega-1)\dot{\varphi} + (2\omega-1)\varphi = 0, \\ \dot{\psi} - \frac{\hbar^2}{2m}\left(\frac{\alpha^2}{(\omega-1)^2} + \frac{\beta^2}{\omega^2} + \gamma^2\right) = 0. \end{cases}$$

In this case we obtain the following invariant solution of system (1):

$$u = C_1|t(t-1)|^{-\frac{1}{2}}\exp\left(\frac{\alpha x_1}{t-1} + \frac{\beta x_2}{t} - \gamma x_3\right),$$
$$v = \frac{m x_1^2}{2(t-1)} + \frac{m x_2^2}{2t} + \frac{\hbar^2}{2m}\left(-\frac{\alpha^2}{t-1} - \frac{\beta^2}{t} + \gamma^2 t\right) + C_2.$$

4.3. $F_3: \ u = \exp\left(\frac{\alpha x_1}{t} - x_2\right)\varphi(\omega), \qquad v = \frac{m x_1^2}{2t} + \psi(\omega), \qquad \omega = t,$

$$2\omega\dot{\varphi} + \varphi = 0, \qquad \dot{\psi} - \frac{\hbar^2}{2m}\left(\frac{\alpha^2}{\omega^2} + 1\right) = 0.$$

The corresponding invariant solution is of the form:

$$u = \frac{C_1}{\sqrt{t}}\exp\left(\frac{\alpha x_1}{t} - x_2\right), \qquad v = \frac{m x_1^2}{2t} + \frac{\hbar^2(t^2-\alpha^2)}{2mt} + C_2.$$

4.4. $F_4: \ u = \exp(-x_1)\varphi(\omega), \qquad v = \psi(\omega), \qquad \omega = t,$

$$\dot{\varphi} = 0, \qquad \dot{\psi} - \frac{\hbar^2}{2m} = 0.$$

The corresponding invariant solution is

$$u = C_1 \exp(-x_1), \qquad v = \frac{\hbar^2}{2m} t + C_2.$$

4.5. $F_5: \ u = \exp(\beta t)\varphi(\omega), \qquad v = \alpha t + \psi(\omega), \qquad \omega = x_1.$

The corresponding reduced system is

$$\begin{cases} 2\beta m\varphi + 2\dot{\varphi}\dot{\psi} + \varphi\ddot{\psi} = 0, \\ 2\alpha m\varphi + \varphi\dot{\psi}^2 - \hbar^2\ddot{\varphi} = 0. \end{cases}$$

4.6. $F_6: \ u = \exp(\alpha t)\varphi(\omega), \qquad v = mtx_1 - \frac{m}{3}t^3 + \psi(\omega), \qquad \omega = t^2 - 2x_1.$

The corresponding reduced system is

$$\begin{cases} \alpha m\varphi + 4\dot{\varphi}\dot{\psi} + 2\varphi\ddot{\psi} = 0, \\ -m^2\omega\varphi + 4\varphi\dot{\psi}^2 - 4\hbar^2\ddot{\varphi} = 0. \end{cases}$$

4.7. $F_7: \ u = \exp\left(\lambda t - \alpha x_3 - \gamma \arctan\frac{x_1}{x_2}\right)\varphi(\omega),$

$$v = -\beta \arctan\frac{x_1}{x_2} + \delta t + \psi(\omega), \qquad \omega = x_1^2 + x_2^2.$$

The corresponding reduced system is

$$\begin{cases} (\lambda m + \beta\gamma)\varphi + 4\omega\dot{\varphi}\dot{\psi} + 2\varphi\dot{\psi} + 2\omega\varphi\ddot{\psi} = 0, \\ (\beta^2 - \hbar^2(\alpha^2\omega + \gamma^2)) + 2\delta m\omega)\varphi - 4\hbar^2\omega\dot{\varphi} - 4\hbar^2\omega^2\ddot{\varphi} + 4\omega^2\varphi\dot{\psi}^2 = 0. \end{cases}$$

4.8. $F_8: \ u = \exp(\beta t)\varphi(\omega), \qquad v = \alpha t + \psi(\omega), \qquad \omega = x_1^2 + x_2^2 + x_3^2.$

The corresponding reduced system is

$$\begin{cases} m\beta\varphi + 4\omega\dot{\varphi}\dot{\psi} + 3\varphi\dot{\psi} + 2\omega\varphi\ddot{\psi} = 0, \\ \alpha m\varphi + 2\omega\varphi\dot{\psi}^2 - \hbar^2(3\dot{\varphi} + 2\omega\ddot{\varphi}) = 0. \end{cases}$$

4.9. $F_9: \ u = \exp(\alpha t - \beta x_2)\varphi(\omega)$,

$$v = m\left(tx_1 - \frac{t^3}{3}\right) + \psi(\omega), \qquad \omega = t^2 - 2x_1.$$

The corresponding reduced system is

$$\begin{cases} \alpha m\varphi + 4\dot\varphi\dot\psi + 2\varphi\ddot\psi = 0, \\ -(m^2\omega + \hbar^2\beta^2)\varphi + 4\varphi\dot\psi^2 - 4\hbar^2\ddot\varphi = 0. \end{cases}$$

4.10. $F_{10}: u = \exp(\beta t - x_2)\varphi(\omega), \qquad v = \alpha t + \psi(\omega), \qquad \omega = x_1.$

The corresponding reduced system is

$$\begin{cases} 2\beta m\varphi + 2\dot\varphi\dot\psi + \varphi\ddot\psi = 0, \\ (2\alpha m - \hbar^2)\varphi + \varphi\dot\psi^2 - \hbar^2\ddot\varphi = 0. \end{cases}$$

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