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Application of Differential Forms to Construction of Nonlocal Symmetries

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Abstract

Differential forms are used for construction of nonlocal symmetries of partial differential equations with conservation laws. Every conservation law allows to introduce a nonlocal variable corresponding to a conserved quantity. A prolongation technique is suggested for action of symmetry operators on these nonlocal variables. It is shown how to introduce these variables for the symmetry group to remain the same. A new hidden symmetry and corresponding group-invariant solution are found for gas dynamic equations.

1 Integrable forms and nonlocal symmetries

Let us consider a system of differential equations

$$F^k(x,t,\vec{u}_x,\vec{u}_t\,..)=0,\ \ k=1,2..m \tag{1}$$

with independent variables x,t and differential variables u^i $(i=1,2..n)$. We call a differential form $\omega=\alpha(x,t,\vec{u}_x,\vec{u}_t\,..)dx+\beta(x,t,\vec{u}_x,\vec{u}_t\,..)dt$ integrable (on (1)) if $D_t(\alpha)=D_x(\beta)$ on (1). (It is suggested that $\omega\not\equiv 0$ on (1).) Here $D_t=\dfrac{\partial}{\partial t}+u^i_t\dfrac{\partial}{\partial u^i}+u^i_{tt}\dfrac{\partial}{\partial u^i_t}+u^i_{tx}\dfrac{\partial}{\partial u^i_x}+\cdots$, $D_x=\dfrac{\partial}{\partial x}+u^i_x\dfrac{\partial}{\partial u^i}+u^i_{tx}\dfrac{\partial}{\partial u^i_t}+u^i_{xx}\dfrac{\partial}{\partial u^i_x}+\cdots$ are operators of full derivatives. The integrability of the form ω guarantees the existence of the function $\Omega(x,t)$ which meets the condition $d\Omega=\omega$. Thus we can extend the system (1) by means of equations $\Omega_x=\alpha,\ \Omega_t=\beta$, with the extended system being consistent. Forms which are integrable on (1) form an infinite-dimensional linear space.

For example, the system of gas dynamic equations in Euler coordinates

$$\rho_t+u\rho_x+u_x\rho=0,\ \ u_t+uu_x+\frac{1}{\rho}p_x=0,\ \ p_t+\gamma pu_x+up_x=0 \tag{2}$$

has the following basis of integrable forms: $\omega_1=\rho u dx-(p+\rho u^2)dt$, $\omega_2=\left(\dfrac{p}{\gamma-1}+\dfrac{\rho u^2}{2}\right)\times dx-\left(\dfrac{\gamma pu}{\gamma-1}+\dfrac{\rho u^3}{2}\right)dt$, $\omega_E=E\left(\dfrac{\rho}{p^{1/\gamma}}\right)p^{1/\gamma}dx-uE\left(\frac{\rho}{p^{1/\gamma}}\right)p^{1/\gamma}dt$, $\omega_3=\rho(x-tu)dx+(\rho u(tu-x)+tp)dt$, $\omega_F=F_x(x,t)dx+F_t(x,t)dt$, with arbitrary functions $E(s)$, $F(x,t)$.

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For $\gamma = 3$, there are two additional forms $\omega_1^\gamma = (tp + t\rho u^2 - \rho ux)dx + (x(p + \rho u^2) - t\rho u^3 - \gamma tpu)dt,\ \omega_2^\gamma = (t^2 p + \rho u^2 t^2 - 2\rho uxt + \rho x^2)dx + (2txp + 2tx\rho u^2 - \rho ux^2 - t^2\rho u^3 - \gamma pt^2 u)dt$. A differential form ω_1 corresponds to the law of momentum conservation and allows to rewrite the system (2) in the form

$$\left(\frac{1}{\rho}\right)_t - p\left(\frac{1}{\rho}\right)_r = uu_r,\ u_t - pu_r = -up_r,\ \epsilon_t - p\epsilon_r = -puu_r(\epsilon = p/(\gamma - 1)\rho)$$

with additional equations $x_r = \dfrac{1}{\rho u},\ x_t = \dfrac{p}{\rho u} + u$. (see [1]). The symmetry operator of (2) $t\,\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}$ cannot be extended on r in the form $t\,\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u} + R(x, t, \rho, u, p, r)\dfrac{\partial}{\partial r}$ so that it would be a symmetry operator for this system. Later we will see the reason of this fact.

Let $X = \xi(x, t, u)\dfrac{\partial}{\partial x} + \tau(x, t, u)\dfrac{\partial}{\partial t} + \eta^i(x, t, u)\dfrac{\partial}{\partial u^i}$ be an infinitesimal operator of some one-parameter transformation group acting on $R^{n+2}(x, t, u)$. Similarly to [2], its action may be extended on dx and dt by formulae $X(dx) = D_x(\xi)dx + D_t(\xi)dt = D(\xi) = D(X(x)),\ X(dt) = D_x(\tau)dx + D_t(\tau)dt = D(\tau) = D(X(t))$, where $D = dxD_x + dtD_t$ is the operator of full exterior differentiation (see [3]). The operator extended in that way can act on differential forms. Besides, it commutes with D. This property is analogous to the fact that a Lie derivative commutes with an exterior derivative. Obviously ω is integrable if and only if $D(\omega) = 0$ on (1). If X is a symmetry operator of (1) and ω is integrable, then $D(X(\omega)) = X(D(\omega)) = 0$ on (1) and $X(\omega)$ is integrable too. It is easy to show that if (1) may be written in the form

$$D(\Psi^i) = \Phi^i = 1 \ldots m, \tag{3}$$

where $\Psi^i = A^i(x, t, u..)dx + B^i(x, t, u..)dt,\ \Psi^i = E^i(x, t, u..)dt \wedge dx$, then X is a symmetry operator of (1) if and only if $D(X(\Psi^i)) = X(\Phi^i)$ on (1).

Action of the symmetry group of (2) on integrable forms may be represented in the form of the table:

	ω_1	ω_2	ω_3	ω_E	ω_F	ω_1^γ	ω_2^γ
$\frac{\partial}{\partial x}$	0	0	ω_s	0	ω_{F_x}	$-\omega_1$	$2\omega_3$
$\frac{\partial}{\partial t}$	0	0	$-\omega_1$	0	ω_{F_t}	$2\omega_2$	$2\omega_1^\gamma$
$t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$	ω_s	ω_1	0	0	ω_{tF_x}	$-\omega_3$	0
$\rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}$	ω_1	ω_2	ω_3	ω_{E_1}	0	ω_1^γ	ω_2^γ
$x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$	ω_1	ω_2	$2\omega_3$	ω_E	$\omega_{xF_x+tF_t}$	$2\omega_1^\gamma$	$3\omega_2^\gamma$
$x\frac{\partial}{\partial x} - 2\rho\frac{\partial}{\partial \rho} + u\frac{\partial}{\partial u}$	0	ω_2	0	ω_{E_2}	ω_{xF_x}	ω_1^γ	ω_2^γ
$xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - t\rho\frac{\partial}{\partial \rho} + (x - tu)\frac{\partial}{\partial u} - 3tp\frac{\partial}{\partial p}$	ω_3	$-\omega_1^\gamma$	0	0	$\omega_{xtF_x+t^2F_t}$	$-\omega_2^\gamma$	0

Here ω_{E_1} and ω_{E_2} are forms of the ω_E type with $E_1(s) = ((\gamma - 1)s\dfrac{d}{ds}E(s) + E(s))/\gamma$ $E_2(s) = E(s) - 2s\dfrac{d}{ds}E(s)$. The form ω_{F_x} equals $d(F_x)$ and so on. The last two columns and the last row arise where $\gamma = 3$. Thus, a linear space of integrable forms is decomposed into a direct sum of two invariant subspaces $V = V_F \oplus \overline{V}$ of forms $\omega_F = dF(x, t)$ and $\overline{\omega} = c^1\omega_1 + c^2\omega_2 + c^3\omega_3 + \omega_E + \{k^1\omega_1^\gamma + k^2\omega_2^\gamma\}$, respectively. $\overline{V}$ contains a finite-dimensional invariant space of forms $\omega = c^1\omega_1 + c^2\omega_2 + c^3\omega_3 + c^4\omega_s + \{k^1\omega_1^\gamma + k^2\omega_2^\gamma\}$. It is convenient to consider integrable forms up to forms which are equal to zero on solutions of (1). Since the space of such forms is invariant under action of a symmetry operator, this action is correctly defined on the factor space W of integrable forms by forms equal to zero on (1).

Theorem 1 *Let W_{inv} be a finite-dimensional invariant subspace of W with the basis $\widetilde{\omega}_1,\ \widetilde{\omega}_2..\widetilde{\omega}_k$, X be a symmetry operator of (1), $X(\omega_i) = c_i^j\omega_j + \omega_{i0}$, where $\omega_i = \alpha_i dx + \beta_i dt$ is the original of $\widetilde{\omega}_i$ under factorization, $\omega_{i0} = 0$ on (1). Then the system (1) extended by*

$$q_{ix} = \alpha_i,\ q_{it} = \beta_i\ i = 1...k \tag{4}$$

has the symmetry operator $Y = X + Q_i\dfrac{\partial}{\partial q_i}$ with $Q_i = c_i^j q_i$.

Proof. Admissibility of Y is equivalent to $D_x(q_i) - q_{ix}D_x(\xi) - q_{it}D_x(\tau) = Y(\alpha_i)\ D_t(q_i) - q_{ix}D_t(\xi) - q_{it}D_t(\tau) = Y(\beta_i)$ on (1). That may be rewritten in the form $c_i^j\alpha_j - \alpha_i D_x(\xi) - \beta_i D_x(\tau) = Y(\alpha_i)\ c_i^j\beta_j - \alpha_i D_t(\xi) - \beta_i D_t(\tau) = Y(\beta_i)$. Multiplying the first equality by dx and the second by dt and summing them together, we arrive at $Y(\omega_i) = c_i^j\omega_j$ which must be fulfilled on (1), i.e., $Y(\omega_i) = c_i^j\omega_j + \omega_{i0}$.

This proof is valid in the case when $Y(\omega_i) = D(f_i) + \omega_{i0}$, where f_i is a function of $x, t, u, q, u_x..$, and the form ω_{i0} is equal to zero on the extended system. In this case, $Q_i = f_i$.

Now it is clear why the operator $X = t\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}$ cannot be retained as a symmetry operator of (2) extended by $r : dr = \omega_1$. The form ω_1 is not eigen for $X : X(\omega_1) = \omega_s$. The space spanned by ω_1, ω_s is invariant under action of the symmetry algebra of (2), when $\gamma \neq 3$. Thus, all symmetry operators may be extended on r, m (here $D(r) = \omega_1$, $D(m) = \rho dx - \rho u dt$). For example, $t\ \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u} \longrightarrow t\ \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u} + m\dfrac{\partial}{\partial r}$, since $X(\omega_s) = 0$.

2 Examples

Extensions of (1) considered above turned out to be useful for construction of quasilocal symmetries proposed in [4]. The authors of [4] consider the sequence of equations

$$w_t = H(w_{xx}), \tag{5}$$

$$v_t = h(v_x)v_{xx}, \tag{6}$$

$$u_t = (h(u)u_x)_x \tag{7}$$

(where $H(s) = \int h(s)ds$), connected with Backlund transforms

$$w_x = v,\ w_t = H(v_x) \qquad v_x = u,\ v_t = h(u)u_x. \tag{8}$$

Then quasilocal symmetry of (6) associated with local symmetry of (5) may be regarded as symmetry of (5) extended on the new variable v. To prolong the action of operator is to extend it on w_x and replace it by v. But quasilocal symmetry of (5) associated with local symmetry of (6) cannot be obtain in a similar way. In [4], the so-called "transition formulas" are used for this purpose.

Here we suggest to use the above procedure of prolongation of the symmetry operator of (6) on the new variable w defined by $dw = vdx + H(v_x)dt$. In [4] it was shown that (6) has a symmetry operator of rotation $X = -v\partial_x + x\partial_v$ when $h(\xi) = 1/(1+\xi^2)$. We have $X(D(w)) = X(vdx + \arctan(v_x)dt) = D\left(\dfrac{x^2 - v^2}{2} + t\right)$. So $X = -v\partial_x + x\partial_v +$

$\left(\frac{x^2-v^2}{2}+t\right)\partial_w$ is the symmetry operator of (6) extended with the first pair of equations (8). Substituting v with w_x, we get the Lie-Backlund symmetry operator of (5).

When $H(\xi)=-3\xi^{-1/3}$, equation (7) has the symmetry operator $X=x^2\partial_x-3xu\partial_u$ (see [4]). Since $X(D(v))=X(udx+h(u)u_xdt)=-D(xv)+vdx-3v_x^{-1/3}dt+x(v_x^{-4/3}v_{xx}-v_t)dt$, we have a quasilocal symmetry operator of (6) $X=x^2\partial_x+(w-xv)\partial_v$, where $dw=vdx-3v_x^{-1/3}dt$. It can be prolonged on w as follows $X=x^2\partial_x+(w-xv)\partial_v+xw\partial_w$ since, $X(dw)=D(xw)+v_x^{-4/3}(w_x-v)dt$. This operator can be treated as a symmetry operator of (6) extended with the first pair of equations (8).

Gas dynamic equations in mass Lagrange coordinates

$$\left(\frac{1}{\rho}\right)_t-u_m=0,\ u_t+p_m=0,\ p_t+\gamma\rho p u_m=0 \tag{9}$$

(where the Euler coordinate x is connected with m by $dx=\frac{dm}{\rho}+udt$) admit a symmetry operator $X=t\partial_u-m\partial_p+\frac{\rho m}{p}\partial_p$ when $\gamma=-1$ (Chaplygin gas) (see [4]). Even extended by $m_x=\rho,\ m_t=-\rho u$, system (2) does not allow this symmetry. The cause of this fact is that $X(dx)=X\left(\frac{dm}{\rho}+udt\right)=tdt-\frac{m}{\rho p}dm=tdt-\frac{m}{\rho p}(\rho dx-\rho udt)=\left(t+\frac{mu}{p}\right)dt-\frac{m}{p}dx$ is not a differential of any function of t,x,m,ρ,u,p. Introducing $Q: D(Q)=\left(t+\frac{mu}{p}\right)dt-\frac{m}{p}dx$, we retain this symmetry of (2) extended by equations $m_x=\rho,\ m_t=-\rho u,\ Q_x=-\frac{m}{p},\ Q_t=t+\frac{mu}{p}$ since $X(D(Q))=0$. The prolonged operator is $t\partial_u-m\partial_p+\frac{\rho m}{p}\partial_p+Q\partial_x$.

3 New hidden symmetry of gas dynamic equations

For the nonlocal variables m and r of Eqs.(2), we have $m_t=-r_x$. It allows to introduce the variable $w: w_x=-m,\ w_t=r$. Then $\rho=m_x=-w_{xx},\ u=r_x/m_x=-w_{tx}/w_{xx}$, $p=-r_t-m_x(r_x/m_x)^2=-w_{tt}+w_{tx}^2/w_{xx}$. So all dependent variables of (2) are expressed in terms of derivatives of w. Equations for ρ, u are fulfilled for any $W(x,t)$ and the equation for p gives

$$\begin{aligned} &w_{ttt}w_{xx}^3-3w_{ttx}w_{xt}w_{xx}^2-w_{txx}(\gamma w_{tt}w_{xx}^2-(\gamma+3)w_{xt}^2w_{xx})+\\ &\qquad w_{xxx}(-(\gamma+1)w_{xt}^3+\gamma w_{tt}w_{xt}w_{xx})=0. \end{aligned} \tag{10}$$

The symmetry algebra of this equation is generated by the following operators

$$X_1=\partial_t,\quad X_2=\partial_x,\quad X_3=t\partial_t,\quad X_4=t\partial_x,\quad X_5=x\partial_x,\quad X_6=w\partial_w,$$

$$Y_1=w\partial_x,\quad Y_2=x\partial_w,\quad Y_3=t\partial_w,\quad Y_4=\partial_w.$$

These operators can be prolonged on derivatives of w. After substitution

$$w_x=-m,\ w_t=r,\ w_{xx}=-\rho,\ w_{xt}=\rho u,\ w_{tt}=-p-\rho u^2,$$

this algebra can be regarded as a symmetry algebra of (2) extended on some of the variables r,m,w defined by

$$m_x=\rho,\ m_t=-\rho u,\ r_x=\rho u,\ r_t=-p-\rho u^2,\ w_x=-m,\ w_t=r.$$

We have $X_1 \to \partial_t,\ X_2 \to \partial_x,\ X_3 \to t\partial_t - u\partial_u - 2p\partial_p,\ X_4 \to t\partial_x + \partial_u,\ X_5 \to x\partial_x - 2\rho\partial_\rho + u\partial_u,\ X_6 \to \rho\partial_\rho + p\partial_p,\ Y_1 \to w\partial_x + (r - mu)\partial_u + 3m\rho\partial_\rho + mp\partial_p + mr\partial_r + m^2\partial_m,\ Y_2 \to x\partial_w - \partial_m,\ Y_3 \to t\partial_w + \partial_r,\ Y_4 \to \partial_w$.

Operators X_i are well-known point symmetry operators of (2). Operators Y_2, Y_3, Y_4 are trivial. The operator Y_1 generates the nontrivial nonlocal symmetry group of (2) which cannot be found in both Euler and Lagrange mass coordinates. Note that for the gas dynamic equations in mass Lagrange coordinates, it is not necessary to introduce w to retain this operator.

The corresponding group-invariant solution of (2) can be written down in the form

$$p = \frac{\beta}{(2\delta)^{1/2}} \left(x_0 - x + ct - \frac{\beta t^2}{2} \right)^{-1/2},$$

$$\rho = \frac{\delta}{(2\delta)^{3/2}} \left(x_0 - x + ct - \frac{\beta t^2}{2} \right)^{-3/2},$$

$$u = c - \beta t,$$

where β, δ, c, x_0 are constant. Since $\beta > 0$, this solution exists only for a finite period of time.

When $\gamma = 3$, the symmetry algebra of (10) is extended by $X_7 = t^2\partial_t + tx\partial_x + tw\partial_w$, which gives the well–known operator $xt\partial_x + t^2\partial_t - t\rho\partial_\rho + (x - tu)\partial_u - 3tp\partial_p$. It is interesting that the case $\gamma = -1$ is not classificating for (10), but $\gamma = 0$ gives extra operators $Z_1 = t^2\partial_x,\ Z_2 = t^2\partial_w,\ Z_3 = \frac{t^2}{2}\partial_t + tx\partial_x + tw\partial_w$.

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