

Relativistic Two-Body Problem: Existence and Uniqueness of Two-Sided Solutions to Functional-Differential Equations of Motion

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Abstract

We study a class of explicitly Poincare-invariant equations of motion (EMs) of two point bodies with a finite speed of propagation of interactions (combination of retarded and advanced ones) that may be considered as functional-differential equations or differential equations with deviating argument of a neutral type. Under conditions having a clear physical interpretation it is proved that there exist ordinary differential equations with all weakly-relativistic solutions satisfying the initial EMs. The existence and uniqueness of two-sided solutions of initial EMs on the infinite time interval are investigated.

1 Introduction

The systems of point particles with a finite speed of propagation of interactions have – in general case – an infinite number of degrees of freedom. This means that the trajectories of particles cannot be specified only by their initial positions and velocities. However, some isolated few-body systems do not obey this rule. The physical and mathematical aspects of this situation have been discussed in [1]–[12]. The main questions arising here are related to the existence and uniqueness properties of two-sided solutions of the differential equations with deviating arguments. These questions, known from the theory of delay-differential equations [13],[14], are significantly obscured by a special structure of the equations of motion in relativistic dynamics, and it is important that the restrictions imposed on functional classes of solutions to provide the uniqueness be physically reasonable.

In this paper some new results in this direction are presented for Poincare-invariant equations of motion with respect to the two-particle trajectories $x_p^\mu := x_p^\mu(t_p) = \{t_p, \mathbf{x}_p(t_p)\}$, $p = 1, 2$, $\mu = 0, 1, 2, 3$. In particular, we extend the results of [11], [12] on more general two-body equations of motion including the advanced-retarded interactions and outline a new proof of the results [4], [8], [9] for one-dimensional symmetric motions.

2 The equations of motion

The equations studied have the following form of a functional differential system (FDS)

$$m_p A_p^\mu = \int ds_q G_{pq} [F_{pq}^\mu(R_{pq}^\alpha, U_p^\alpha, U_q^\alpha) + f_{pq}^\mu(R_{pq}^\alpha, U_p^\alpha, U_q^\alpha, A_p^\alpha, A_q^\alpha)], \quad p, q = 1, 2; p \neq q; (1)$$

where $\{R_{pq}^\alpha\} = \{x_p^\alpha - x_q^\alpha\}$, $U_p^\mu = dx_p^\mu/ds_p$, $A_p^\alpha = dU_p^\mu/ds_p$, $ds_p = \sqrt{1 - \dot{\mathbf{x}}_p^2}$, $U_p^\mu U_{p\mu} = 1$; $F_{pq}^\mu U_{p\mu} = 0$, $f_{pq}^\mu U_{p\mu} = 0$ (conditions to preserve the normalization of U_p^μ), the scalar products are taken and the indexes are raised and lowered with respect to the Minkowski metric $diag(1, -1, -1, -1)$;

$$G_{pq} = [g \theta(t_p - t_q) + g^* \theta(t_q - t_p)] \delta[R_{pq}^\alpha R_{pq\alpha}],$$

θ is the Heaviside function, δ is the Dirac function, g and g^* are interaction constants, $m_p > 0$ are particle masses.

The functions F_{pq}^μ are constructed from the Poincare-invariant combinations of the four-dimensional vectors $\{R_{pq}^\alpha\} = \{t_p^\alpha - t_q^\alpha, \mathbf{x}_p(t_p) - \mathbf{x}_q(t_q)\}$, $\{U_p^\alpha\} = \{U_p^0, \mathbf{U}_p\}$, and f_{pq}^μ is constructed from the four-vectors R_{pq}^α , U_p^α , $A_p^\alpha = \{A_p^0, \mathbf{A}_p\}$, ($p, q = 1, 2$). Here the boldface letters denote three-dimensional (spatial) parts of four-vectors. We assume the units with the speed of light equal to 1.

We suppose that

- (a) for $R_{pq}^\alpha \neq 0$ the functions F_{pq}^μ and f_{pq}^μ are homogeneous rational expressions of R_{pq}^α of the order -1 and 0, respectively, and they do not involve any dimensional parameters;
- (b) these functions are analytic in $\mathbf{U}_1, \mathbf{U}_2$ in the neighborhood of the point $\mathbf{U}_1 = \mathbf{0}$, $\mathbf{U}_2 = \mathbf{0}$;
- (c) f_{pq}^μ is a linear homogeneous function of A_1^α, A_2^α ;
- (d) for $\mathbf{U}_1 = \mathbf{0}, \mathbf{U}_2 = \mathbf{0}$ the spatial parts of F_{pq}^μ agree with the Newtonian limit:

$$\mathbf{F}_{pq} = (\mathbf{x}_p - \mathbf{x}_q) / |x_p^0 - x_q^0|^2. \quad (2)$$

In special cases FDS (1) reduce , e.g., to the equations of motions proposed by A. Poincare in his famous paper [15] or to the equations arising in any linear massless field theory in the Minkowski space [16]. The other example is given by the equations resulting from the one-particle action functionals [11],[12]

$$\int ds_p \left\{ -m_p - \frac{g}{2\pi} \int ds_q G_{pq} \widetilde{f}_{pq}(U_p^\alpha U_{q\alpha}) \right\} \quad (3)$$

where $\widetilde{f}(z)$ is supposed to be analytic in the neighborhood of $z = 1$. For $\widetilde{f}(z) = z$ under appropriate choice of g_{pq} and g_{pq}^* , this action yields the equations of motion of a charged particle in classical electrodynamics [17].

Eqs. (1) are considered from the infinite past; this agrees with the physical notion about an isolated particle system. The r.h.s. of FDS (1) is defined on the trajectories from $C^2(\mathbf{R}, \mathbf{R}^3)$ satisfying

$$|\dot{\mathbf{x}}_p(t)| < 1, p = 1, 2; \quad \mathbf{x}_1(t) \neq \mathbf{x}_2(t). \quad (4)$$

We say that the trajectory $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is a solution of the system (1) if the equations (1) and conditions (4) are satisfied for all $t \in \mathbf{R}$.

3 Existence and uniqueness of two-sided solutions

Define a domain $\mathbf{D}(\varepsilon) \subset \mathbf{R}^{12}$ of the variables $\mathbf{v}_p, \mathbf{x}_p$, $p = 1, 2$, by the relation

$$\mathbf{v}_1^2 + \mathbf{v}_2^2 + k/|\mathbf{x}_1 - \mathbf{x}_2| \leq \varepsilon,$$

where $k = (|g| + |g^*|)/\min(m_1, m_2)$. This is a domain where the classical energy of particles is small as compared with their self-energy.

Let $\mathbf{W}(\varepsilon)$ be a class of weakly-relativistic trajectories defined by

$$\mathbf{x}_p \in C^2(\mathbf{R}, \mathbf{R}^3); \{\dot{\mathbf{x}}_p(t), \mathbf{x}_p(t)\} \in \mathbf{D}(\varepsilon) \text{ for } \forall t \in \mathbf{R}; \sup\{|\ddot{\mathbf{x}}_p(t)|, t \in \mathbf{R}\} < \infty.$$

Now the question is as follows: when can the solutions of the functional-differential system (1) be described by solutions of the ordinary differential system (ODS)

$$\ddot{\mathbf{x}}_p(t) = \mathbf{H}_p(\dot{\mathbf{x}}_1(t), \dot{\mathbf{x}}_2(t), \mathbf{x}_1(t), \mathbf{x}_2(t)) \quad (5)$$

The following two theorems generalize some of the results of the papers [11],[12] using the methods described in [12].

Theorem 1. *Let the conditions (a)–(d) for the r.h.s. of (1) be satisfied. Then there exists a value $\varepsilon > 0$ and the Lipschitz continuous functions $\mathbf{H}_p : \mathbf{D}(\varepsilon) \rightarrow \mathbf{R}^3$, $p = 1, 2$, such that any trajectory $\mathbf{x}_p \in C^2(\mathbf{R}, \mathbf{R}^3)$, $\{\dot{\mathbf{x}}_p(t), \mathbf{x}_p(t)\} \in \mathbf{D}(\varepsilon)$ for $\forall t \in \mathbf{R}$ satisfying ODS (5) on \mathbf{R} is a solution of FDS (1).*

The functions \mathbf{H}_p are obtained by the certain iteration procedure [12] which is convergent in the weakly-relativistic region. The space of solutions of FDS (1) is much wider than that of (5) and in fact the ODS (5) selects a finite-parametric family of them. One may impose some restrictions on the solutions of (1) to select those trajectories that are described by (5). Under these restrictions we expect that the solution of FDS (1) can be specified uniquely by pointwise “initial” conditions

$$\begin{aligned} \dot{\mathbf{x}}_p(t_0) &= \mathbf{v}_{0p}, & \mathbf{x}_p(t_0) &= \mathbf{x}_{0p}; \\ |\mathbf{v}_{0p}| &< 1, & \mathbf{x}_{01} &\neq \mathbf{x}_{02}; & p &= 1, 2. \end{aligned} \quad (6)$$

Theorem 2. *Let the conditions (a)–(d) for the r.h.s. of (1) be satisfied. Let $g > 0$, $g^* \geq 0$. Then there exist the values $\varepsilon > 0$, $\varepsilon' > 0$ and Lipschitz continuous functions $\mathbf{H}_p : \mathbf{D}(\varepsilon) \rightarrow \mathbf{R}^3$, $p = 1, 2$, such that for any $\{\mathbf{v}_{0p}, \mathbf{x}_{0p}\} \in \mathbf{D}(\varepsilon')$*

- (i) *there is a unique solution of (1) satisfying (6) in the class $\mathbf{W}(\varepsilon)$;*
- (ii) *any solution of (1) from $\mathbf{W}(\varepsilon')$ satisfies the ordinary differential system (5).*

Thus we see that under clear physical restrictions the ODS (5) represents all the weakly relativistic solutions of FDS (1).

From this theorem it follows that the Poincare-invariance properties of the system (7) are transferred to the system (5). This leads to the well-known Currie-Hill differential conditions of Poincare invariance [18], [19] for the functions \mathbf{H}_p .

4 One-dimensional symmetric two-body problem

Now we shall use the results obtained to study the one-dimensional symmetric motion of two charged particles of the same sign in ordinary classical electrodynamics with the retarded interaction [2],[4],[9] and the analogous delayed-advanced model problem of Fokker-Wheeler-Feynman electrodynamics [7],[8],[10].

Both cases, first considered by R.D.Driver, are represented by the following system

$$\frac{\ddot{x}(t)}{(1-\dot{x}^2(t))^{3/2}} = \frac{k}{r^2(t)} \frac{1-\dot{x}(t-r)}{1+\dot{x}(t-r)} + \frac{k^*}{q^2(t)} \frac{1+\dot{x}(t+q)}{1-\dot{x}(t+q)}, \quad (7)$$

$$r(t) = x(t) + x(t-r), \quad q(t) = x(t) + x(t+q).$$

where

(c) $k > 0, k^* = 0$ in case of purely retarded interaction and

(d) $k = k^* > 0$ in case of retarded-advanced interaction.

We say that $x(t)$ is a solution of (7) if $x \in C^2(\mathbf{R}, \mathbf{R}_+)$

$$\dot{x}(t) < 1, \quad x(t) > 0 \quad (8)$$

(that corresponds to (4)) and $x(t)$ satisfies (7) $\forall t \in \mathbf{R}$.

In the one-dimensional case the conditions (6) at the moment t_0 have the form

$$\dot{x}(t_0) = v_0, \quad x(t_0) = x_0; \quad (9)$$

$$|v_0| < 1, \quad x_0 > 0.$$

In this case $\mathbf{D}(\varepsilon) = \{(v, x) \in \mathbf{R}^2 : x > 0, v^2 + (k + k^*)/(2x) \leq \varepsilon\}$

The interesting feature of the equations (7) is that the restrictions on the functional class of solutions that select a unique solution under the conditions (9) can be reduced to a minimum according to (8). In fact any solution of (7) satisfying (9) appears to be a trajectory from $\mathbf{W}(\varepsilon)$. Here the following property plays the decisive role.

Lemma (Driver[2],[7], Hoag & Driver [10], Zhdanov [4],[8], [9]). *Let $x(t)$ satisfies (7),(8),(9) where $v_0^2 + (k + k^*)/x_0 \leq \varepsilon$. Then $\dot{x}^2(t) + (k + k^*)/(2x(t)) \leq \varepsilon N(\varepsilon)$, and $N_2(\varepsilon) \leq 4 \ddot{x}(t)x^2(t)/(k + k^*) \leq N_3(\varepsilon)$, where $N_i(\varepsilon) = O(1)$ for $\varepsilon \rightarrow 0$, $i = 1, 2, 3$.*

The statement of this lemma relies completely on the estimates of [2], [4], [9] in case of (c) and on the estimates of [7], [10] in the case (d).

REMARK. Some intensification of this result yields $N_i(\varepsilon) \rightarrow 1$ for $\varepsilon \rightarrow 0$.

Because the equation (7) is a special case of (1), we derive the existence of a one-dimensional version of (5):

$$\ddot{x}(t) = H(\dot{x}(t), x(t)). \quad (10)$$

Now, in view of the Lemma, in order to investigate the two-sided solutions of (7)–(9), one may either use Theorems 1 and 2 directly or use the equation (10). In the latter case, first, one should use the results of [7], [4], [9] to state the existence and uniqueness of solutions of the problem (7–9) for the data with $v_0 = 0$ (for sufficiently small ε).

The method how to extend these results on $v_0 > 0$ has been outlined in [4], [8], [9] by using some monotonicity properties of the solutions. Here, to extend the domain of the

initial data on values $v_0 > 0$, we use the equation (10) in the following way. Suppose (from the contrary) the existence of two solutions $x(t)$ and $x'(t)$ with the same initial data for sufficiently small ε (without restricting the sign of v_0). From the Lemma and on account of the results of [2],[7], [10], [4],[8], [9], it follows that the solutions intersect the axis $v = 0$ in the $v - x$ plane. Then in view of the uniqueness for $v_0 = 0$ we see that $x(t)$ and $x'(t)$ are solutions of ordinary differential equation (10) with the Lipschitz continuous r.h.s., and, because they pass through the same point of the phase space and the function $\dot{x}(t)$ is monotonous, we have $x(t) \equiv x'(t)$. This result may be formulated as the following.

Theorem 3. *Let either assumption (c) or (d) be valid. Then there exist the values $\varepsilon > 0$, $\varepsilon' > 0$ and Lipschitz continuous function $H : \mathbf{D}(\varepsilon) \rightarrow \mathbf{R}$, such that for any $(v_0, x_0) \in \mathbf{D}(\varepsilon')$*

- (i) *there is a unique solution of the problem (7)–(9);*
- (ii) *any solution of this problem satisfies the ordinary differential equation (10).*

This theorem covers all weakly-relativistic region of (v_0, x_0) . The following statement involves also a part of the ultra-relativistic region in case of purely retarded interactions.

Theorem 4. *Under the assumption (c) there exists $\varepsilon_0 > 0$ such that for (v_0, x_0) , $v_0 < 0$, satisfying the inequality $k(1 - v_0^2)^{1/2}/x_0 < \varepsilon_0$, there is a unique solution of the problem (7)–(9).*

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