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Nonlocal Symmetry of Nonlinear Wave Equations

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Abstract

A class of nonlinear wave equations is considered. Symmetry of these equations is extended using nonlocal transformations.

Let us consider the nonlinear wave equations

$$L_1(u) \equiv u_{00} - \partial_1 [c(u)] = 0, \tag{1}$$

where $c(u)$ are arbitrary smooth functions,

$$u_\mu = \partial_\mu u = \frac{\partial u}{\partial x_\mu}, \qquad u_{\mu\nu} = \partial_\mu \partial_\nu u = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu} \quad (\mu, \nu = 0, 1).$$

We extend a symmetry of Eq.(1) using the nonlocal transformation, which connects it with the corresponding linear equation [1]. The nonlocal ansatze and nonlinear superposition formulae are obtained.

1. Nonlocal ansatze. The group classification of Eq.(1) was fulfiled in [2].

1. $c(u)$ — arbitrary: $< P_0, P_1, D_1 >$, where

$$P_0 = \partial_0, \quad P_1 = \partial_1, \quad D_1 = x_0\partial_0 + x_1\partial_1;$$

2. $c(u) = k_1 \exp(k_2 u)$: $\quad < P_0, P_1, D_1, Q >$, where

$$Q = k_2 x_1 \partial_1 + 2\partial_u;$$

3. $c(u) = k_1 u^{k_2} \quad < P_0, P_1, D_1, D_2 >$, where

$$D_2 = k_2 x_1 \partial_1 + 2u\partial_u;$$

4. $c(u) = k_1 u^{-4/3} \quad < P_0, P_1, D_1, D_2, \Pi_1 >$, where

$$\Pi_1 = -x_1^2 \partial_1 + 3x_1 u \partial_u;$$

5. $c(u) = k_1 u^{-4} \quad < P_0, P_1, D_1, D_2, \Pi_2 >$, where

$$\Pi_2 = x_0^2 \partial_0 + x_0 u \partial_u.$$

We obtain the coresponding Lie ansatze in a standard way. We have

1. $u = \varphi(\omega), \quad \omega = x_0^{-1}x_1;$

2. $u = \varphi(\omega) - 2(c+1)\ln x_0, \quad \omega = x_1 x_0^c;$

3. $u = x_0^{-2\frac{c+1}{a}}\varphi(\omega), \quad \omega = x_1 x_0^c;$

4. a) $c = 1$

$$u = (x_1^2+1)^{-3/2}\exp\left\{\frac{3}{2}\mathrm{arth}\, x_1\right\}\varphi(\omega),$$
$$\omega = \arctan x_1 - \ln x_0;$$

b) $c = -1$

$$u = (x_1^2-1)^{-3/2}\exp\left\{-\frac{3}{2}\arctan x_1\right\}\varphi(\omega),$$
$$\omega = \mathrm{arth}\, x_1 + \ln x_0;$$

c) $c = 0$

$$u = x_1^{-3}\exp\left\{-\frac{3}{2}x_1^{-1}\right\}\varphi(\omega),$$
$$\omega = \ln x_0 + x_1^{-1};$$

5. a) $c = 1$

$$u = (x_0^2+1)^{1/2}\exp\left\{-\frac{1}{2}\arctan x_1\right\}\varphi(\omega),$$
$$\omega = \arctan x_1 - \ln x_0;$$

b) $c = -1$

$$u = (x_0^2-1)^{1/2}\exp\left\{\frac{1}{2}\mathrm{arth}\, x_1\right\}\varphi(\omega),$$
$$\omega = \mathrm{arth}\, x_1 + \ln x_0;$$

c) $c = 0$

$$u = x_0\exp\left\{\frac{1}{2x_0}\right\}\varphi(\omega),$$
$$\omega = \ln x_1 + x_0^{-1}.$$

Here a, c are arbitrary constant parameters. Nonlinear Eq.(1) is reduced to the linear equation

$$v_{11} = c(y_1)v_{00}, \quad v = v(y_0, y_1) \tag{2}$$

by the nonlocal transformation [1]

$$u = y_1, \quad x_0 = v_0, \quad x_0 = v_1. \tag{3}$$

Equations (2) were exhaustively investigated from the classical–groups standpoint in [3]. It was obtained there that equations (2) with $c(y_1)$ of the form

$$(by_1^2 + cy_1 + d)^{-2}\exp\left\{2(c-a)\int(by_1^2+cy_1+d)^{-1}\mathrm{d}y_1\right\} \tag{4}$$

admit the four-dimensional Lie group. At the same time, Eq.(1) with the same function $c(u)$, as one can see, admits only the three-dimensional group $< P_0, P_1 >$. In [3] from the equations (2) with function $c(y_1)$ of the form (4) we picked out some canonical forms. For these classes of equations corresponding Lie ansatze were constructed and reduction to ODE was obtained.

From these results via the nonlocal transformation (3) we construct nonlocal ansatze for Eq.(1), which determine the corresponding nonlocal symmetries and reduced ODE:
1. $c(u) = (u^2+1)^{-2} \exp\{-4a \arctan u\}$:

a) $$\begin{aligned}
&x_0 = (u^2+1)^{1/2} \exp\{(s+2a)\arctan u\}\,\varphi(\omega),\\
&x_1 = (u+s)(u^2+1)^{-1/2}\exp\{s\arctan u\}\,\varphi(\omega) + 2a\tau x_0(u^2+1)^{-1}\dot{\varphi}(\omega),\\
&\omega = \tau\exp\{2a\arctan u\};\\
&x_1(u^2+1) - 2ax_0\partial_1^{-1}u_0 - (u+s)\left(x_0\partial_1^{-1}u_0 + x_1u - \partial_1^{-1}u\right) = 0;\\
&(4a^2\omega^2-1)\varphi + 4a(a+s)\omega\dot{\varphi} + (1+s^2)\varphi = 0.
\end{aligned}$$

b) $$\begin{aligned}
&x_0 = (u^2+1)^{1/2}\exp(a\arctan u)\left\{(4a^2\tau+2a)\exp(2a\arctan u)\,\varphi(\omega)+\right.\\
&\left.4a^2\tau\left[\omega+(1+2a\tau)\exp(2a\arctan u)\right]\exp(2a\arctan u)\,\dot{\varphi}(\omega)\right\},\\
&x_1 = (a+u)(u^2+1)^{-1/2}\exp(a\arctan u)\left[\omega+(1+2a\tau)\exp(2a\arctan u)\right]\varphi(\omega)+\\
&4a^3\tau^2(u^2+1)^{-1/2}\exp(3a\arctan u)\left[\omega+(1+2a\tau)\exp(2a\arctan u)\right]\dot{\varphi}(\omega),\\
&\omega = 2a^2\tau^2\exp(2a\arctan u);\\
&x_1(\partial_1^{-1}u_0)(1+u^2) - \frac{1}{4a}x_0\left[4a^2(\partial_1^{-1}u_0)^2 + \exp(-4a\arctan u) - 1\right] -\\
&\left[(\partial_1^{-1}u_0)(a+u)+a\right](x_0(\partial_1^{-1}u_0) + x_1u - \partial_1^{-1}u) = 0;\\
&4a^2(\omega^2-1)\varphi + 8a^2(1+s)\omega\dot{\varphi} + \left[1+a^2(1+2s)^2\right]\varphi = 0.
\end{aligned}$$

2. $c(u) = (1-u)^{-2(1+a)}(1+u)^{-2(1-a)}$:

a) $$\begin{aligned}
&x_0 = (1-u^2)^{1/2}\left(\frac{1-u}{1+u}\right)^{a+s}\varphi(\omega),\\
&-x_1 = u(1-u^2)^{-1/2}\left(\frac{1-u}{1+u}\right)^{s}\varphi + (1-u^2)^{1/2}\frac{2s}{(1+u)^2}\left(\frac{1-u}{1+u}\right)^{s-1}\varphi+\\
&2\tau a(1-u^2)^{1/2}\left(\frac{1-u}{1+u}\right)^{a+s-1}\frac{1}{(1+u)^2}\dot{\varphi}(\omega),\\
&\omega = \tau\left(\frac{1-u}{1+u}\right)^{s};\\
&x_1(u^2-1) - 2ax_0\partial_1^{-1}u_0 - (u+2a)(x_0\partial_1^{-1}u_0 + x_1u - \partial_1^{-1}u) = 0,\\
&(4a^2\omega^2-1)\varphi + 4a(a+2s)\omega\dot{\varphi} + (4s^2-1)\varphi = 0.
\end{aligned}$$

b) $$\begin{aligned}
&x_0 = (1-u^2)^{1/2}\left(\frac{1-u}{1+u}\right)^{3a/2}\left[4a^2\tau+2a\right]\varphi(\omega)+\\
&4a^2\tau(1-u^2)^{1/2}\left(\frac{1-u}{1+u}\right)^{3a/2}\left[(1+2a\tau)\left(\frac{1-u}{1+u}\right)^{a}+\omega\right]\dot{\varphi}(\omega),\\
&-x_1 = \left\{u(1-u^2)^{-1/2}\left(\frac{1-u}{1+u}\right)^{a/2} + (1+u^2)^{1/2}\frac{a}{(1+u)^2}\left(\frac{1-u}{1+u}\right)^{a/2-1}\right\}\times\\
&\left[(1+2a\tau)\left(\frac{1-u}{1+u}\right)^{a}+\omega\right] + 2\frac{(1-u^2)^{1/2}}{(1+u)^2}\left(\frac{1-u}{1+u}\right)^{a/2}\times
\end{aligned}$$

$$\Big\{a(1+2a\tau)\left(\frac{1-u}{1+u}\right)^{a-1}+2a^3\tau^2\left(\frac{1-u}{1+u}\right)^{a-1}-\frac{1}{2}a\left(\frac{1-u}{1+u}\right)^{a-1}+$$
$$\frac{1}{2}a\left(\frac{1-u}{1+u}\right)^{-a-1}\Big\}\varphi(\omega)+\frac{(1-u^2)^{1/2}}{(1+u)^2}\left(\frac{1-u}{1+u}\right)^{a/2}\times$$
$$\Big[(1+2a\tau)\left(\frac{1-u}{1+u}\right)^{a}+\omega\Big]\Big\{(4a^3\tau^2-a)\left(\frac{1-u}{1+u}\right)^{a-1}+$$
$$a\left(\frac{1-u}{1+u}\right)^{-a-1}\Big\}\dot\varphi(\omega),$$
$$\omega=2a^2\tau^2\left(\frac{1-u}{1+u}\right)^{a}-\frac{1}{2}\left[\left(\frac{1-u}{1+u}\right)^{a}+\left(\frac{1-u}{1+u}\right)^{-a}\right];$$
$$x_1(u^2-1)\partial_1^{-1}u_0+\frac{1}{4a}x_0\left[1-4a^2(\partial_1^{-1})^2-\left(\frac{1-u}{1+u}\right)^{-2a}\right]-$$
$$\Big[(a+u)\partial_1^{-1}u_0+a\Big]\Big(x_0\partial_1^{-1}u_0+x_1u\partial_1^{-1}u\Big)=0,$$
$$4a^2(\omega^2-1)\varphi+8a^2(s+1)\omega\dot\varphi+\left[a^2(2s+1)^2-1\right]\varphi=0.$$

3. $c(u)=u^{-4}\exp\left(-\frac{2}{u}\right)$:

a) $$x_0=u\exp\left(\frac{s+1}{u}\right)\varphi(\omega),$$
$$x_1=(1-\tfrac{s}{u})\exp\left(\tfrac{s}{u}\right)\varphi(\omega)-\frac{\tau}{u}\exp\left(\frac{s+1}{u}\right)\dot\varphi(\omega),$$
$$\omega=\tau\exp\left(\frac{1}{u}\right);$$
$$x_1u^2+x_0\partial_1^{-1}u_0-(u-s)\Big(x_0\partial_1^{-1}u_0+x_1u-\partial_1^{-1}u\Big)=0,$$
$$(\omega^2-1)\ddot\varphi+(2s+1)\omega\dot\varphi+s^2\varphi=0.$$

b) $$x_0=-2u\exp\left(\frac{1}{2u}\right)\exp\left(-2s\tau\exp\left(\frac{1}{u}\right)\omega^{-1}\right)\varphi(\omega)\left[s\exp\left(\frac{1}{u}\right)\omega^{-1}-\right.$$
$$\left.2s\tau^2\omega^{-2}\exp\left(\frac{2}{u}\right)\right]+2\tau u\exp\left(\frac{3}{2u}\right)\exp\left(-2s\tau\exp\left(\frac{1}{u}\right)\omega^{-1}\right)\dot\varphi(\omega),$$
$$x_1=\left[1-\frac{1}{2u}\right]\exp\left(-2s\tau\exp\left(\frac{1}{u}\right)\omega^{-1}\right)\varphi(\omega)+u\exp\left(\frac{1}{2u}\right)\times$$
$$\left\{\frac{2s\tau}{u^2}\exp\left(-\frac{1}{u}\right)\omega^{-1}+2\tau s\exp\left(-\frac{1}{u}\right)\omega^{-2}\left[\frac{\tau^2}{u^2}\exp\left(\frac{1}{u}\right)+\frac{1}{u^2}\exp\left(-\frac{1}{u}\right)m\right]\right\}\times$$
$$\exp\left(-2s\tau\exp\left(\frac{1}{u}\right)\omega^{-1}\right)\varphi(\omega)-u\left[\frac{\tau^2}{u^2}\exp\left(\frac{1}{u}\right)+\frac{1}{u^2}\exp\left(-\frac{1}{u}\right)m\right]\times$$
$$\exp\left(\frac{1}{2u}\right)\exp\left(-2s\tau\exp\left(\frac{1}{u}\right)\omega^{-1}\right)\dot\varphi(\omega),$$
$$\omega=\tau^2\exp\left(\frac{1}{u}\right)-\exp\left(-\frac{1}{u}\right);$$
$$2x_1u\partial_1^{-1}u_0+x_0\left(\left(\partial_1^{-1}u_0\right)^2+\exp\left(-\frac{2}{u}\right)\right)-$$
$$\Big[(2u-1)\partial_1^{-1}u_0+2s\Big]\Big(x_0\partial_1^{-1}u_0+x_1u\partial_1^{-1}u\Big)=0,$$
$$4\omega^2\ddot\varphi+8\omega\dot\varphi+(1-16s^2\omega^{-2})\varphi=0,\ \Big(\partial_1^{-1}u=\textstyle\int u\,dx_1\Big).$$

2. The formula of nonlocal superposition and generating solutions. The above-mentioned nonlocal transformation of variables (3) and the linear superposition principle for the solutions of Eq.(2) allow us to construct the corresponding nonlinear superposition principle for Eq.(1).

Theorem. *The superpositon formula for solutions of Eq.(1)*

$$\overset{(k)}{u}(x_0, x_1), \quad (k = 1, 2)$$

has the form

$$\overset{(3)}{u}(x_0, x_1) = \overset{(1)}{u}\left(\overset{(1)}{\tau}\right)\frac{d\overset{(1)}{\tau_1}}{dx_1} + \overset{(2)}{u}\left(\overset{(2)}{\tau}\right)\frac{d\overset{(2)}{\tau_1}}{dx_1},$$

$$\overset{(1)}{\tau} + \overset{(2)}{\tau} = x, \ x = (x_0, x_1), \ \overset{(k)}{\tau} = \left(\overset{(1)}{\tau_0}, \overset{(1)}{\tau_1}\right),$$

$$\overset{(1)}{u}\left(\overset{(1)}{\tau}\right) = \overset{(2)}{u}\left(\overset{(2)}{\tau}\right), (k = 1, 2),$$

$$\overset{(1)}{u_0}\left(\overset{(1)}{\tau}\right) d\overset{(1)}{\tau_1} = \overset{(2)}{u_0}\left(\overset{(2)}{\tau}\right) d\overset{(2)}{\tau_1}.$$

Here $\overset{(k)}{u}(x_0, x_1)$*,* $k = 1, 2$*, are the known solutions of Eq.(1) and* $\overset{(3)}{u}(x_0, x_1)$ *is a new solution,*

$$\overset{(k)}{u_0}\left(\overset{(k)}{\tau}\right) \equiv \partial_{\overset{(k)}{\tau_0}}\overset{(k)}{u}\left(\overset{(k)}{\tau}\right).$$

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