Linear Complexity of Generalized Hall's Sextic Sequence

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Abstract—Let p be a prime with $p \equiv 1 \pmod{6}$. In this paper, we calculate the linear complexity of the connection polynomial of a balanced binary Hall's sextic sequence. It shows that the sequence has large linear complexity and balanced properties.

Keywords—generalized cyclotomic sequence, autocorrelation value, linear complexity

I. INTRODUCTION

For cryptographic applications, pseudo-random sequences are required to have the property of unpredictability. Balance and linear complexity are two main components that indicate this feature. A lot of authors investigate the minimal polynomials and the linear complexity of cyclotomic sequences (see [2,4,5,6,7,9]). In 1997, Ding [2] calculated the linear complexity of generalized cyclotomic binary sequences of order 2. In 2001, Kim and Song [7] got the linear complexity of cyclotomic binary sequences of order 6. Recently, Edemskii [4] obtained partial exponential sums of cyclotomic classes of order 4 and 6 in an extension over $\text{GF}(2)$ to compute the linear complexity of binary sequences and [5] gave a method to calculate the linear complexity of generalized cyclotomic binary sequences of period p^n .

Also some applications, such as communication systems and cryptography require a sequence is easy to distinguish from a time-shifted version of itself, which is to say: have low autocorrelation. So there are several authors investigate difference sets and low correlation values of binary sequences (see [1,2,3,8]). Liu et al. [10] computed the autocorrelation values and the linear complexity of DH generalized cyclotomic sequences of period p^n over $\text{GF}(q)$ provided that $p = qf + 1$ and q are primes, which generalized Sidelnikov's results [11].

A sequence $(S = s_0, s_1, \dots, s_{N-1}, \dots)$ is said to be N -periodic if $s_i = s_{i+N}$ for all $i \geq 0$. The linear complexity of a sequence S over $\text{GF}(2)$ is an important characteristic of its equality. It is defined to be the smallest positive integer L for which there exist constants $c_1, \dots, c_L \in \text{GF}(2)$ such that

$$s_g = c_1 s_{g-1} + c_2 s_{g-2} + \dots + c_L s_{g-L} \text{ for all } g \geq L.$$

We call $f(x) = 1 - c_1 x - \dots - c_L x^L$ the connection polynomial.

Let $p = 6f + 1$ be an odd prime, where f is a positive integer. Let θ be a primitive root of p . Set $D = (\theta^6)$ a cyclic subgroup of Z_p^* . There are six classes $D_i = \theta^i D$, $i = 0, 1, 2, 3, 4, 5$, and $Z_p^* = \bigcup_{i=1}^5 D_i$.

Define $c_0 = D_0 \cup D_1 \cup D_3$, $c_1 = D_2 \cup D_4 \cup D_5 \cup \{0\}$, the generalized Hall's sextic sequence SSS is defined by

$$s_i = \begin{cases} 1, & \text{if } i \pmod{p} \in c_0, \\ 0, & \text{if } i \pmod{p} \in c_1, \end{cases} \text{ for } i \geq 0. \quad (1.1)$$

Define

$$S(x) = s_0 + s_1 x + \dots + s_{N-1} x^{N-1} = \sum_{i \in \Omega} x^i \quad (1.2)$$

as the characteristic polynomial of the sequence S . It is well-known that the connection polynomial of the binary sequence S of period N is given by

$$f_s(x) = \frac{1 - x^N}{\gcd(1 - x^N, S(x))},$$

where the constant coefficient of $\gcd(1 - x^N, S(x))$ is 1. So the linear complexity of SSS is given by

$$L = N - \deg(\gcd(1 - x^N, S(x))).$$

In this paper, we will compute the connection polynomial and the linear complexity of the sequences (1.1).

II. LINEAR COMPLEXITY

Let $p = 6f + 1$ be a prime. There is a decomposition over Z : $p = A^2 + 3B^2$, $A \equiv 1 \pmod{3}$. We define the sign of B as follows: if $B \equiv 0 \pmod{3}$, then $B/3 \equiv \text{ind}_\theta 3 \pmod{3}$ and always $B \equiv -\text{ind}_\theta 2 \pmod{3}$, where $i = \text{ind}_\theta 2 \pmod{3}$ is the smallest positive integer such that $\theta^i = 2 \pmod{p}$.

We define the polynomial

$$S_j(x) = \sum_{i \in D_j} x^i, j = 0, 1, 2, 3, 4, 5.$$

Let α be a primitive P th root of unity in an extension of $\text{GF}(2)$. Define

$$S = (S_0(\alpha), S_1(\alpha), S_2(\alpha), S_3(\alpha), S_4(\alpha), S_5(\alpha)).$$

Let $p = A^2 + 3B^2$, $A \equiv 1(\text{mod } 3)$. By [7], we have the following results.

$$\text{ind}_\theta 2 \equiv 0(\text{mod } 3), \quad \text{that} \quad \text{is,}$$

$$A = 4 + 6a, B = 3 + 6b, \text{ then}$$

$$(1) S = (1, 1, 0, 0, 1, 0) \quad \text{if} \quad a \equiv 1(\text{mod } 2), b \equiv 0(\text{mod } 2);$$

$$(2) S = (1, 0, 1, 0, 0, 1) \quad \text{if} \quad a \equiv 1(\text{mod } 2), b \equiv 1(\text{mod } 2);$$

$$(3) S = (\gamma, 0, 1, 1, -\gamma, 0, 1) \quad \text{where } \gamma \text{ is a root of the equation } x^2 + x + 1 = 0 \text{ and } b \equiv 1(\text{mod } 2);$$

$$(4) S = (\gamma, 1, 0, 1, -\gamma, 1, 0) \quad \text{if } a \equiv 1(\text{mod } 2), b \equiv 1(\text{mod } 2).$$

$$\text{If } f \equiv 0(\text{mod } 2), A = 1 + 6a, B = 6b, \text{ then}$$

$$(5) S = (1, 0, 0, 0, 0, 0) \text{ if } b \equiv 0(\text{mod } 2);$$

$$(6) S = (1, 1, 1, 0, 1, 1) \text{ if } b \equiv 1(\text{mod } 2);$$

$$(7) S = (\gamma, 0, 0, 1, -\gamma, 0, 0) \text{ if and}$$

$$b \equiv 1(\text{mod } 2);$$

$$(8) S = (\gamma, 1, 1, 1, -\gamma, 1, 1) \text{ if } a \equiv 1(\text{mod } 2), \text{ and } b \equiv 1(\text{mod } 2).$$

Now we compute the linear complexity of the sequence defined as (1.1). Then by Blahut's theorem the linear complexity of the sequence S defined as (1.5) is

$$L = p - |\{t \mid S(\alpha^t) = 0, t = 0, 1, \dots, p-1\}|, \quad (2.3)$$

where $S(x) = s_0 + s_1x + \dots + s_{p-1}x^{p-1}$ is the characteristic polynomial of the sequence S . So the linear complexity of the sequence S reduces to counting the number of roots of $S(x)$ in the set $\{\alpha^t \mid t = 0, 1, \dots, p-1\}$.

It is clear that the characteristic polynomial of the sequence S in (1.1) is $S(x) = S_0(x) + S_1(x) + S_3(x)$.

Now we compute the linear complexity L and the connection polynomial $f_S(x)$ of the Hall's sextic cyclotomic sequence. Let α be a p -th primitive root of unity in an extension field over $\text{GF}(2)$. Set

$$d_i(x) = \prod_{l \in D_i} (1 - \alpha^{-l}x), i = 0, 1, 2, 3, 4, 5.$$

Theorem 2.1. Let $p = 6f + 1$ be a prime, $p = A^2 + 3B^2$,

$$A, B \in \mathbb{Z}, A \equiv 1(\text{mod } 3).$$

$$\text{if } A = 4 + 6a, B = 3 + 6b, a, b \in \mathbb{Z}.$$

$$(1) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 0(\text{mod } 2), \text{ then}$$

$$f_S(x) = (1-x)d_5(x), L = (p-5)/6.$$

$$(2) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 1(\text{mod } 2), \text{ then}$$

$$f_S(x) = (1-x)d_0(x)d_1(x)d_3(x)d_4(x)d_5(x), L = (5p+1)/6.$$

$$(3) \text{ If } a \equiv 0(\text{mod } 2) \text{ and } b \equiv 0(\text{mod } 2), \text{ then}$$

$$f_S(x) = 1 - x^p, L = p.$$

$$(4) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 1(\text{mod } 2), \text{ then}$$

$$f_S(x) = (1-x)d_2(x)d_5(x), L = (p+2)/3.$$

$$\text{If } f = 2, A = 1 + 6a, B = 6b, a, b \in \mathbb{Z}.$$

$$(5) \text{ If } a \equiv 0(\text{mod } 2) \text{ and } b \equiv 0(\text{mod } 2), \text{ then}$$

$$f_S(x) = d_0(x)d_3(x)d_5(x), L = (p-1)/2.$$

$$(6) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 1(\text{mod } 2), \text{ then}$$

$$f_S(x) = d_1(x)d_4(x)d_5(x), L = (p-1)/2.$$

$$(7) \text{ If } a \equiv 0(\text{mod } 2) \text{ and } b \equiv 1(\text{mod } 2), \text{ then}$$

$$f_S(x) = d_0(x)d_2(x)d_3(x)d_5(x), L = (2p-2)/3.$$

$$(8) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 1(\text{mod } 2), \text{ then}$$

$$f_S(x) = d_1(x)d_2(x)d_4(x)d_5(x), L = (2p-2)/3.$$

Proof. If then it is clear that

$$S(1) = \frac{p-1}{2} = 3f = 1(\text{mod } 2).$$

$$(1) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 0(\text{mod } 2). \text{ If } a \in D_i, \text{ then}$$

$$S(\alpha^i) = S_0(\alpha^i) + S_1(\alpha^i) + S_3(\alpha^i) = S_i(\alpha) + S_{1+i}(\alpha) + S_{3+i}(\alpha). \text{ Hence by Lemma 1, we know that } S(\alpha^i) = 0$$

If

$$i \in D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4, \quad S(\alpha^i) \neq 0 \quad \text{otherwise.}$$

Then

$$\gcd(1-x^p, S(x)) = d_0(x)d_1(x)d_2(x)d_3(x)d_4(x),$$

$$\text{so } f_S(x) = (1-x)d_5(x) \text{ and } L = (p-5)/6.$$

$$(2) \text{ If } a \equiv 1(\text{mod } 2) \text{ and } b \equiv 1(\text{mod } 2). \text{ Similarly, we have that } S(\alpha^i) = 0$$

$$\text{If } i \in D_2, \quad S(\alpha^i) \neq 0 \quad \text{otherwise. Then}$$

$$\gcd(1-x^p, S(x)) = d_2(x), \text{ so}$$

$$f_S(x) = (1-x)d_0(x)d_1(x)d_3(x)d_4(x)d_5(x),$$

$$\text{and } L = (5p+1)/6.$$

(3) If $a \equiv 0 \pmod{2}$ and $b \equiv 0 \pmod{2}$. Similarly, we have that $S(\alpha^i) \neq 0$ for any $i \in Z_p$, then

and $L = p$.

(4) If $a \equiv 0 \pmod{2}$ and $b \equiv 1 \pmod{2}$. Similarly, we have that $S(\alpha^i) = 0$ if $i \in D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4$,

$S(\alpha^i) \neq 0$ otherwise. Then
 $f_s(x) = (1-x)d_2(x)d_5(x)$,
and $L = (p+2)/3$.

If $f \equiv 0 \pmod{2}$, then it is clear that $S(1) = \frac{p-1}{2} = 0 \pmod{2}$.

(5) If $a \equiv 0 \pmod{2}$ and $b \equiv 0 \pmod{2}$. Similarly, we have that $S(\alpha^i) = 0$ if $i \in D_1 \cup D_2 \cup D_4 \cup \{0\}$,

$S(\alpha^i) \neq 0$ otherwise. Then
 $f_s(x) = d_0(x)d_3(x)d_5(x)$,
and $L = (p-1)/2$.

(6) If $a \equiv 1 \pmod{2}$ and $b \equiv 0 \pmod{2}$. Similarly, we have that $S(\alpha^i) = 0$ if $i \in D_0 \cup D_2 \cup D_3 \cup \{0\}$,

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