On Symmetry Reduction of Nonlinear Generalization of the Heat Equation

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Abstract

Reductions and classes of new exact solutions are constructed for a class of Galilei-invariant heat equations.

It is well-known that the $n$-dimensional linear heat equation

$$ku_t = u_{11} + \ldots + u_{nn}$$

(1)

where $u_t = \frac{\partial u}{\partial t}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, is invariant under the extended complete Galilei algebra $\tilde{A}_G(1, n)$. Unfortunately, the equation (1) cannot describe a great number of real processes of heat and mass transfer. The known nonlinear generalization of the equation (1)

$$u_t + \nabla (F(u) \nabla u) = 0$$

(2)

is invariant under the Galilei algebra only if $F(u) = $ const. Galilei-invariant nonlinear generalizations of the equation (1) were described in the paper [1].

Let formulate the necessary results. Consider the equation of the second order

$$u_t + F(t, \mathbf{x}, u, u_1, u_2) = 0,$$

(3)

where $u$ is the set of $s$-th order partial derivatives of $u$ with respect to the space variables $x_1, x_2, \ldots, x_n$ ($s = 1, 2$).

The equation (3) is invariant under the extended classical Galilei algebra $\tilde{A}_G(1, n)$ iff it is of the form

$$u_t + \frac{1}{2m} (\nabla u)^2 + \Phi(<1>, <2>, \ldots, <n>) = 0,$$

(4)

where $\Phi$ is an arbitrary smooth function, $m = $ const.
of the rank \(F\) where \(\omega\) is an arbitrary smooth function. It allows us to use the results of the paper [3].

As in [3], in the present paper we confine ourselves by consideration of such subalgebras which do not contain operator \(M\).

Let \(AO[p, q] = <J_{ab}; a, b = p, \ldots, q>\);

\[
\Phi(d_0, d_1, \gamma_1) = <G_{d_0} + \gamma_1 P_{d_0}, \ldots, G_{d_0} + \gamma_1 P_{d_1} > + AO[d_0, d_1];
\]

AE\((n - k) = <P_{k+1}, \ldots, P_n > + AO[k + 1, n] \ (0 \leq k \leq n - 1);\)

AE\((n - n) = AE(0) = 0;\)

AE\(_1(n - k) = <G_{k+1}, \ldots, G_n > + AO[k + 1, n] \ (0 \leq k \leq n - 1);\)

AE\(_1(n - n) = AE_1(0) = 0.\)

Let \(d_1, \ldots, d_p\) be natural numbers which satisfy the condition \(1 = d_0 < d_1 < \ldots < d_p \leq n\). With respect to \(\hat{G}(1, n)\)-conjugation, the algebra \(AG(1, n)\) contains 6 maximal
subalgebras of the rank $n$. For each of these algebras we show a corresponding ansatz and reduced equation.

1) $AE(n)$: \( u = \varphi(t), \quad \dot{\varphi} + \Phi(0; 0; \ldots; 0) = 0. \)

2) $\Phi(1, d_1, \gamma_1) \oplus \ldots \oplus \Phi(d_{p-1} + 1, d_p, \gamma_p) \oplus AE(n-k)$ \( (d_p = m; 1 \leq k \leq n) : \)
\[
u = \frac{m}{2} \sum_{j=1}^{p} \left( \frac{x_{d_j-1}^2 + \ldots + x_{d_j}^2} {t - \gamma_j} \right) + \varphi(t), \quad \dot{\varphi} + \Phi(m\sigma_1; m^2\sigma_2; \ldots, m^k\sigma_k; 0; \ldots; 0) = 0, \]
where
\[
\sigma_1 = y_1 + y_2 + \ldots + y_k, \\
\sigma_2 = y_1y_2 + y_1y_3 + \ldots + y_{k-1}y_k, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\sigma_k = y_1y_2 \ldots y_k
\]
are the elementary symmetrical polynomials and \( y_1 = \ldots = y_{d_1} = \frac{1}{\omega - \gamma_1}, \)
\( y_{d_1+1} = \ldots = y_{d_2} = \frac{1}{\omega - \gamma_2}, \ldots, y_{d_{p-1}+1} = \ldots = y_{d_p} = \frac{1}{\omega - \gamma_p}. \)

3) \( < T + \alpha M, J_{12} + \beta M > \oplus AE(n-2) \) \((\alpha, \beta \in R) : \)
\[
u = \alpha mt + \beta \arctan \left( \frac{x_1}{x_2} \right) + \varphi(x_1^2 + x_2^2), \]
\[
\alpha m + \frac{1}{2m} (\beta^2 \omega^{-1} + 4\omega \dot{\varphi}^2) + \Phi(4\dot{\varphi} + 4\omega \ddot{\varphi}; 4\dot{\varphi}^2 + 8\omega \ddot{\varphi} - \beta^2 \omega^{-2}; 0; \ldots; 0) = 0.
\]

4) \( < T + \alpha M > \oplus AE(n-1) \) \((\alpha \in R) : \)
\[
u = \alpha mt + \varphi(x_1), \quad \alpha m + \frac{1}{2m} \dot{\varphi}^2 + \Phi(\ddot{\varphi}; 0; \ldots; 0) = 0.
\]

5) \( < T + \alpha G_1 > \oplus AE(n-1) \) \((\alpha > 0) : \)
\[
u = \alpha mtx_1 - \frac{1}{3} \alpha^2 m \beta^3 + \varphi(\alpha t^2 - 2x_1), \quad -\frac{\alpha m}{2} \omega + \frac{2}{m} \dot{\varphi}^2 + \Phi(4\ddot{\varphi}; 0; \ldots; 0) = 0.
\]

6) \( < T + \alpha M > \oplus AO[1, k] \oplus AE(n-k) \) \((\alpha \in R; 3 \leq k \leq n) : \)
\[
u = \alpha mt + \varphi \left( \sum_{i=1}^{k} x_i^2 \right), \quad \alpha m + \frac{2}{m} \omega \dot{\varphi}^2 + \Phi(y_1; \ldots; y_k; 0; \ldots; 0) = 0,
\]
where \( y_p = \frac{2^p(k-1)!}{(k-p)!p!} (\ddot{\varphi})^{p-1} (k\ddot{\varphi} + 2p\omega \ddot{\varphi})(p = 1, \ldots, k). \)

References

