Nonlinear Mathematical Physics 1996, V.3, N 3–4, 319–329.

Discrete Symmetry and Its Use to Find Multi-Soliton Solutions of the Equations of Anisotropic Heisenberg Ferromagnets

N.A. BELOV[†], *A.N. LEZNOV*[‡] *and W.J. ZAKRZEWSKI*[†]

[†]*Institute for Problems in Mechanics,*
Russian Academy of Sciences, Moscow, Russia
[‡]*Institute for High Energy Physics, Protvino, Russia*
[†]*Department of Mathematical Sciences,*
University of Durham, Durham DH1 3LE, England

Abstract

Explicit solutions of the chain equations describing discrete transformations of the Landau-Lifshitz equation are found in the most economical way. This chain can be considered as a generalization of the Toda chain (the case of a rational spectral parameter) to the case of an elliptic curve. We show that the process of deriving multi-soliton solutions of the Landau-Lifshitz equation in an explicit form with help of these results becomes relatively straightforward.

1 Introduction

In one of our previous papers [1], we gave an explicit solution of the chain of discrete transformations for the Landau-Lifshitz (L-L) equation describing a classical anisotropic Heisenberg ferromagnet. Our solution was derived by direct, although rather complicated, calculations.

In another paper [2], we have found and investigated solutions of some chains of equations. These equations, in our opinion, play the fundamental role in the problems connected with elliptic curves in the same sense as the equations of the Toda chain play in the case of rational curves [3].

One of the aims of this paper is to use the results of [2] to derive multi-soliton solutions of the L-L equation in the most economic way.

Let us add that this problem has been studied before. In fact in [4], this problem was considered from the point of the Lax pair representation, while in [5] it was discussed on the basis of the Hamiltonian formalism.

2 L–L equation and its discrete transformation

For completeness, let us start by repeating the statment of the problem as given in [1].

The L–L equation arises out of the generalization of the Heisenberg model of isotropic ferromagnets to an anisotropic case in the classical region. In this formulation, which is close to the original [6], the L-L equation describes the evolution of a unit vector field $\vec{S}$ as a function of time t and one space variable x

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \vec{S} \times (\hat{J}\vec{S}), \quad \hat{J} = \mathrm{diag}(J_1, J_2, J_3),$$

where indices t and x denote the time and the space derivatives and constants J_n are related to the moments of inertia of a nonaxisymmetric "rigid body".

It is convenient to perform a stereographic projection and so to introduce complex fields u and v

$$u = \frac{S_1 + iS_2}{1 + S_3}, \quad v = \frac{S_1 - iS_2}{1 + S_3}.$$

Then, disregarding the "reality" condition $u^* = v$, the equation can be represented as a system

$$\begin{aligned} &\imath u_t + u_{xx} - \frac{2v}{1+uv}(u_x^2 + R(u)) + R_u(u) = 0, \\ &-\imath v_t + v_{xx} - \frac{2u}{1+uv}(v_x^2 + R(v)) + R_v(v) = 0, \end{aligned} \tag{1}$$

where

$$R(u) = \alpha u^4 + \gamma u^2 + \alpha, \quad \alpha = \frac{J_2 - J_1}{4}, \quad \gamma = \frac{J_1 + J_2}{2} - J_3.$$

System (1) is invariant with respect to the following discrete nonlinear transformation $u \to U$, $v \to V$:

$$U = \frac{1}{v}, \quad \frac{1}{1+UV} - \frac{1}{1+uv} = \frac{vv_{xx} - v_x^2 + \alpha(v^4 - 1)}{(v_x)^2 + R(v)}.$$

This transformation plays the key role in our work. In fact, we will use it in the following way. Instead of solving the original equations (1), we will consider the transformation as an iterative procedure for generating from one set of functions u and v another one. Then, having "solved" this iterative procedure, we will start from a given solution of (1) and will generate many other solutions, among which we will find the ones which satisfy the conjugation condition.

Thus, if we denote u and v as u_n and v_n, and U and V as u_{n+1} and v_{n+1}, respectively, we will have a chain

$$u_{n+1} = \frac{1}{v_n}, \quad \frac{1}{1+u_{n+1}v_{n+1}} - \frac{1}{1+u_n v_n} = \frac{(\ln v_n)_{xx} + \alpha(v_n^2 - v_n^{-2})}{(\ln v_n)_x^2 + \alpha(v_n^2 + v_n^{-2}) + \gamma} \tag{2}$$

which, in what follows, we will call *the L-L chain.*

Looking at (2) and its inverse form

$$v_{n-1} = \frac{1}{u_n}, \quad \frac{1}{1+u_{n-1}v_{n-1}} - \frac{1}{1+u_n v_n} = \frac{(\ln u_n)_{xx} + \alpha(u_n^2 - u_n^{-2})}{(\ln u_n)_x^2 + \alpha(u_n^2 + u_n^{-2}) + \gamma},$$

we observe that, in general, the chain is infinite except when for some u_0 or/and v_r $(0 < r)$:

$$u_{0x}^2 + R(u_0) = 0, \qquad v_{rx}^2 + R(v_r) = 0.$$

In this case, we cannot find v_{-1} or/and v_{r+1}, and we have the interrupted L-L chain from left or/and right ends.

Similar problems have been studied before. Solutions of the corresponding discrete chains for many integrable systems are given in [7]. All the chains discussed there are closelly related to the Toda chain. The L-L chain is more complicated and, as we will show, leads to the Toda chain only in a some limiting case.

3 Solution of the linear problem satisfying a boundary condition at the left end of the L-L chain

Here we will discuss the constraints to be satisfied by the solutions u_0, v_0 of system (1) which arise when we impose the boundary conditions on (2).

Let

$$u_{0x}^2 + R(u_0) = 0. \tag{3}$$

Then, the first equation in (1) is satisfied if $u_{0t} = 0$. Next we rewrite the second equation introducing new variables and a new unknown function: $s = \ln u_0$ and $q = (u_0 v_0 - 1)/(u_0 v_0 + 1)$. Then, $q(t, s)$ has to satisfy

$$\imath q_t + [P(q' + q^2) - Qq]' = 0,$$

where

$$P = \gamma + 2\alpha \cosh 2s, \qquad Q \equiv \frac{P'}{2} = 2\alpha \sinh 2s,$$

and the prime denotes (here and in what follows) the derivative with respect to s. Next we introduce a function ϕ given by

$$\phi' = q, \qquad -\imath\phi_t = P(q' + q^2) - Qq.$$

This allows us to reduce the problem to a linear equation for the function $y = \exp\phi$:

$$\imath y_t + Py'' - Qy' = 0. \tag{4}$$

Note that for any solution of (4), we take $q = (\ln y)'$ and find that

$$u_0(s) = e^s, \qquad v_0(s) = e^{-s}\frac{1 + (\ln y)'}{1 - (\ln y)'}.$$

Here $s(x)$ is the elliptic function (sn), which satisfies the equation

$$s_x^2 + P(s) = 0,$$

which follows from (3).

Note also that separating off the time variable in (4) by puting $y_t = \imath\lambda^2 y$, we obtain an ordinary differential equation

$$Ly \equiv Py'' - Qy' = \lambda^2 y, \tag{5}$$

which possesses the following first integral

$$I(y) \equiv \left(\lambda - \frac{\delta_1\delta_2}{P}\right) y'^2 - 2\frac{\lambda Q}{P} yy' - \lambda \left(\frac{\lambda}{P} - 1\right) y^2 = \text{const}, \tag{6}$$

where $\delta_{1,2} = \gamma \pm 2\alpha$ (note, for testing, that $\delta_1\delta_2 = Q^2 - P(Q' - P)$). The general solution of (5) has a form

$$y = C_1 y_1 + C_2 y_2, \qquad y_{1,2} \equiv y^\epsilon = \sqrt{\lambda P - \delta_1\delta_2}\,\exp(\epsilon J), \quad \epsilon = \pm 1,$$

$$J = \Delta \int \frac{\sqrt{P} ds}{\lambda P - \delta_1\delta_2}, \quad \Delta = \sqrt{(\lambda - \delta_1)(\lambda - \delta_2)\lambda} \quad (\text{const} = -4C_1C_2\Delta^2),$$

where J with help of substitution $\xi = \cosh s$ may be represented as a sum of two elliptic integrals of the first and third kinds [9]. Underline that $I(y_1) = I(y_2) = 0$.

For the further discussion, it is useful to define also *a primitive solution* of (4) for a fixed value of λ as that

$$y^\epsilon_\lambda = e^{\imath\lambda t} y^\epsilon_\lambda. \tag{7}$$

A consequence of this definition is that: $L y^\epsilon_\lambda = \lambda y^\epsilon_\lambda, \quad I(y^\epsilon_\lambda) = 0$.

4 Solution of the semi-infinite L-L chain

Here, for completeness, we present again the results first given in [2]

4.1 Statement of the problem

We want to find a solution of the L-L chain starting from its left end:

$$u_1 = e^s, \qquad v_1 = e^{-s}\frac{1 + (\ln Y)'}{1 - (\ln Y)'},$$

where Y is an arbitrary function of s. We rewrite (2) as a chain only for unknowns v_n $\left(\text{excluding } u_n \text{ by help of equality } u_n = \frac{1}{v_{n-1}}\right)$

$$\frac{1}{v_{n+1} + v_n} + \frac{1}{v_n + v_{n-1}} = \frac{(\ln F_n)_x}{2v_{nx}} \equiv \frac{(\ln F_n)'}{2v'_n}, \tag{8}$$

where

$$F_n = R(v_n) + v^2_{nx} = R(v_n) - P v'^2_n.$$

Then boundary conditions take a form

$$v_0 = e^{-s}, \qquad v_1 = v_0 \frac{Y + Y'}{Y - Y'}. \tag{9}$$

4.2 Elliptic Toda I chain

Next we introduce the functions ρ_n by

$$\exp \rho_n = \frac{(v_n)_x^2 + R(v_n)}{(v_{n-1} + v_n)(v_n + v_{n+1})}. \tag{10}$$

Then, it is not difficult to convince oneself, by differentiating and using the invariance of (8) with respect to the transformation: $v_n \to \dfrac{1}{v_n}$, that the following equations are satisfied

$$\begin{aligned}
(\rho_n)_{xx} &= -\exp \rho_{n-1} + 2 \exp \rho_n - \exp \rho_{n+1} + \alpha(v_{n-1}^2 - 2v_n^2 + v_{n+1}^2), \\
\exp \pi_n &= \frac{v_{n-1} v_{n+1}}{v_n^2} \exp \rho_n, \\
(\pi_n)_{xx} &= -\exp \pi_{n-1} + 2 \exp \pi_n - \exp \pi_{n+1} + \alpha(v_{n-1}^{-2} - 2v_n^{-2} + v_{n+1}^{-2}).
\end{aligned} \tag{11}$$

System (11), called in [2] the first elliptic Toda chain (EToda1), is very useful in the (trigonometric) case $\alpha = 0$, because, in this case, it becomes equivalent to two noninteracting Toda chains. Let us consider this case in detail.

4.3 Trigonometric case

Assuming for simplicity that $\gamma = 1$, we find that $s = \imath x$. We can rewrite equations (11) in terms of the variable s and then exploit the fact that the solution of the usual Toda chain with a fixed end ($\exp \rho_0 = \exp \pi_0 = 0$) is well known [3]:

$$\exp \rho_n = \frac{D_{n-1}(f) D_{n+1}(f)}{D_n(f)^2}, \quad \exp \pi_n = \frac{D_{n-1}(g) D_{n+1}(g)}{D_n(g)^2},$$

where f, g are arbitrary functions and $D_n(f)$ is the n-th order principal minor of the matrix

$$\begin{pmatrix} f & f' & f'' & \cdots \\ f' & f'' & f''' & \\ f'' & f''' & f'''' & \\ \cdots & & & \end{pmatrix}.$$

Then v_n is proportional to the ratio of $D_n(g)$ and $D_n(f)$. Using (9) and a relation which follows from the definition of $\exp \rho_1$, namely,

$$\frac{v_1^2 - v_1'^2}{(v_0 + v_1)(v_1 + v_2)} = \frac{D_2(f)}{f^2},$$

we obtain

$$v_n = e^{-s} \frac{D_n(g)}{D_n(f)}, \quad g = Y + Y', \quad f = Y - Y',$$

which give us a solution of the semi-infinite L-L chain (8)–(9) in this special case. Note that we have reproduced the calculations of the Appendix in [1].

4.4 Elliptic Toda II chain

In the general case, *ie*, when $\alpha \neq 0$ we first seek solutions of (8) in the form

$$v_n = (-1)^n v_0 \frac{1+w_n}{1-w_n}. \tag{12}$$

Then from (8) we get a chain

$$\frac{1}{w_n - w_{n+1}} - \frac{1}{w_{n-1} - w_n} = \frac{A_n}{B_n}, \tag{13}$$

where

$$A_n = (P(w_n' + w_n^2) - Qw_n)', \quad B_n = Pw_n'(w_n' + w_n^2 - 1) + Qw_n(w_n^2 + 1) - Q'w_n^2.$$

The boundary conditions now take the form

$$w_0 = 0, \qquad w_1 = \frac{Y}{Y'}. \tag{14}$$

Using (8) and (10), it is easy to verify that

$$(\ln(v_n - v_0))_{xx} = \Big(1 - \frac{(v_{n+1}+v_0)(v_{n-1}+v_0)}{(v_n - v_0)^2}\Big) \exp\rho_n - \alpha(v_n^2 - v_0^2) - 2\Big(\frac{v_{0x}}{v_n - v_0}\Big)_x.$$

Generalizing, it further we obtain

$$\Big(\ln\frac{v_n \pm v_0}{v_0}\Big)_{xx} = \Big(1 - \frac{(v_{n+1} \mp v_0)(v_{n-1} \mp v_0)}{(v_n \pm v_0)^2}\Big) \exp\rho_n - \alpha(v_n^2 - v_0^2) + \Big(s_x \frac{v_n \mp v_0}{v_n \pm v_0}\Big)_x.$$

Next we define new functions $\theta_n^{\pm}$:

$$\exp\theta_n^{\pm} = \frac{(v_{n+1} \pm (-1)^{n+1} v_0)(v_{n-1} \pm (-1)^{n-1} v_0)}{(v_n \pm (-1)^n v_0)^2} \exp\rho_n.$$

Then from (10)–(12) and recalling the last formulae, we find that $\theta_n^{\pm}$ and w_n satisfy the following system of equations

$$\begin{aligned} (\theta_n^{\pm})_{xx} &= -\exp\theta_{n-1}^{\pm} + 2\exp\theta_n^{\pm} - \exp\theta_{n+1}^{\pm} + (s_x(w_{n-1}^{\pm 1} - 2w_n^{\pm 1} + w_{n+1}^{\pm 1}))_x, \\ \exp\theta_n^- &= \frac{w_{n-1}w_{n+1}}{w_n^2} \exp\theta_n^+. \end{aligned} \tag{15}$$

Note that in terms of w_n: $\exp\theta_n^+ = B_n(w_{n+1} - w_n)^{-1}(w_n - w_{n-1})^{-1}$.

Let us observe that if we consider the following system of equations for the unknown functions a_n and b_n

$$\begin{aligned} \frac{a_{n-1}a_{n+1}}{a_n^2} &= \Big(\frac{s_x b_n - a_{nx}}{a_n}\Big)_x \equiv P\Big(\frac{a_n' - b_n}{a_n}\Big)' + Q\Big(\frac{a_n' - b_n}{a_n}\Big), \\ \frac{b_{n-1}b_{n+1}}{b_n^2} &= \Big(\frac{s_x a_n - b_{nx}}{b_n}\Big)_x \equiv P\Big(\frac{b_n' - a_n}{b_n}\Big)' + Q\Big(\frac{b_n' - a_n}{b_n}\Big), \end{aligned} \tag{16}$$

then equations (15) follow from it due to the following simple relations

$$\exp\theta_n^- = \frac{a_{n-1}a_{n+1}}{a_n^2}, \quad \exp\theta_n^+ = \frac{b_{n-1}b_{n+1}}{b_n^2}, \quad w_n = \frac{a_n}{b_n}.$$

Systems (15) or (16) were called in [2] the second elliptic Toda chain (EToda2). We expect that EToda2 plays the same role in the case of an elliptic curve (the elliptic parametrization of the spectral parameter) as the ordinary Toda chain plays in the case of a rational curve [8].

4.5 Solution of the chain in the general case

Next we impose the following set of boundary conditions at the left end of the chain (16)

$$a_0 = 0, \quad b_0 = 1, \quad a_1 = Y, \quad a_2 = \det \begin{pmatrix} Y & LY \\ Y' & (LY)' \end{pmatrix}. \tag{17}$$

The last condition (instead of b_1) comes (13) for $n = 1$.

As we have shown in [2], the following expression is a solution of (16) and (17)

$$a_n = (AB)^r_n, \quad b_n = (AB)^c_n, \tag{18}$$

where $(AB)^{r(c)}_n$ denote principal n-th order minors of the matrix, which is derived from AB by excluding its first row (column):

$$E = \begin{pmatrix} 0 & Y' & \tilde{Y} & L' & \tilde{L} & L_2' & \cdots \\ Y & L & P\tilde{Y}' & L_2 & P\tilde{L}' & L_3 & \\ Y' & L' & \tilde{L} & L_2' & \tilde{L}_2 & L_3' & \\ L & L_2 & P\tilde{L}' & L_3 & P\tilde{L}_2' & L_4 & \\ L' & L_2' & \tilde{L}_2 & L_3' & \tilde{L}_3 & L_4' & \\ L_2 & L_3 & P\tilde{L}_2' & L_4 & P\tilde{L}_3' & L_5 & \\ \cdots & & & & & & \end{pmatrix}.$$

Here $L_n = \overbrace{LL...L}^{n} Y$, $\quad \tilde{Y} = Y'' - Y$, $\quad \tilde{L}_n = L_n'' - L_n = (\tilde{L})_n$, the operator L is defined by (5).

5 Elliptic Toda II chain with two fixed ends

In this section we want to derive conditions on the initial function Y, under which the solution of the last section will satisfy the additional condition $b_{2n+1} = 0$. For this purpose, it is necessary to consider more carefully the structure of the AB matrix. The elements of this matrix contain only four different functional structures: $L_k, L_k', L_k'' - L_k, P(L_k'' - L_k)'$. Let us calculate them when the initial function Y is chosen as

$$Y = \sum_{k=1}^{2n+1} c_k y_k^{\epsilon_k} = \sum y_{\lambda_k}^{\epsilon_k}, \tag{19}$$

where primitive functions y_k^ϵ are defined by (5) and constants c_k are put in as multipliers.

First we note the following identities, which will be useful in what follows

$$Ly_k = \lambda_k y_k, \quad y_k' = q_k y_k, \quad q_k = \frac{\lambda_k Q + \epsilon_k \sqrt{\Delta(\lambda_k) P}}{P\lambda_k - g_1 g_2},$$
$$\Delta(\lambda) \equiv \lambda(\lambda - g_1)(\lambda - g_2) \tag{20}$$

so $\epsilon = \pm 1$ correspond to the two linear independent solutions of (5).

Using this notation, we have

$$L_r' = \sum c_k q_k y_k \lambda_k^r, \quad L_r'' - L_r = \sum \Big(\frac{\lambda_k}{P} - 1 \Big) y_k \lambda_k^r = -\sum \tilde{q}_k q_k y_k \lambda_k^{r-1},$$

$$P(L_r'' - L_r)' = -\sum \tilde{q}_k \lambda_k^r y_k, \tag{21}$$

where $\tilde{q}_k = \dfrac{\lambda_k Q + \epsilon\sqrt{\Delta(\lambda_k)P}}{P}$. All these identities have a pure algebraic character and can be checked directly from the definitions.

The simplest way to understand the general construction is to consider first the simplest example. Let $n = 1$. Then the condition $b_3 = 0$ together with (18) implies that the determinant of the matrix

$$\begin{pmatrix} \sum q_k y_k & -\sum \lambda_k^{-1} q_k \tilde{q}_k y_k & \sum \lambda_k q_k y_k \\ \sum \lambda_k y_k & -\sum \tilde{q}_k y_k & \sum \lambda_k^2 y_k \\ \sum \lambda_k q_k y_k & -\sum q_k \tilde{q}_k y_k & \sum \lambda_k^2 q_k y_k \end{pmatrix} \tag{22}$$

must vanish. This is because each column of this matrix is a linear combination of the three vectors:

$$t_k = \begin{pmatrix} q_k y_k \\ \lambda_k y_k \\ \lambda_k q_k y_k \end{pmatrix},$$

and so the determinant of the matrix is proportional to the product of the determinant of a matrix constructed from these columns and the determinant of the following matrix

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \dfrac{\tilde{q}_1}{\lambda_1} & \dfrac{\tilde{q}_2}{\lambda_2} & \dfrac{\tilde{q}_3}{\lambda_3} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

Thus, using the definition of $\tilde{q}_k$, we finally obtain the condition for the interruption of the chain at the third step, *ie*, $b_3 = 0$ in the following form

$$\det{}_3 \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \dfrac{\sqrt{\Delta_1}}{\lambda_1} & \dfrac{\sqrt{\Delta_2}}{\lambda_2} & \dfrac{\sqrt{\Delta_3}}{\lambda_3} \end{pmatrix} = 0. \tag{23}$$

From this example, the situation in the case of an arbitrary n is quite clear and as a direct corollary of $b_{2n+1} = 0$, we conclude that we have to set to zero $\det_{2n+1}$ of the matrix, each column of which has the form

$$a_k = \begin{pmatrix} 1 \\ \lambda_k \\ \dfrac{\sqrt{\Delta_k}}{\lambda_k} \\ \lambda_k^2 \\ \sqrt{\Delta_k} \\ \cdots\cdots\cdots \\ \lambda_k^{n-2}\sqrt{\Delta_k} \end{pmatrix}. \tag{24}$$

In the trigonometric case $\alpha = 0$, (24) coincides with the corresponding condition on the parameters λ of [1].

6 Multi-soliton solutions of the L-L equation

Here we use the results of the preceding sections to construct n-soliton solutions of the L–L equation in an explicit form.

To do this, we recall the solution of the L–L equation with $v_n,\ u_n$ given in terms of σ functions (12). Then, given our boundary conditions, we have the following chain of solutions of the L–L equation obtained by the successive applications of the discrete transformation

$$\begin{pmatrix} \exp s \\ \exp -s\dfrac{1+(\ln Y)'}{1-(\ln Y)'} \end{pmatrix} \to \ldots.. \to \begin{pmatrix} \exp s\dfrac{1-\sigma_n}{1+\sigma_n} \\ \exp -s\dfrac{1+\sigma_{n+1}}{1-\sigma_{n+1}} \end{pmatrix} \to \ldots \to \begin{pmatrix} \exp s\dfrac{1-\sigma_{2n}}{1+\sigma_{2n}} \\ \exp -s\dfrac{1+\sigma_{2n+1}}{1-\sigma_{2n+1}} \end{pmatrix}, \quad (25)$$

where $\to$ denotes that we perform our discrete transformation.

Next we observe that if we find such a function Y (all σ_n are expressed in terms of Y and its derivatives) that

$$v^*_{2n+1} = u_1, \quad u^*_{2n+1} = v_1, \quad (26)$$

then in the middle of the chain we have

$$v_{n+1}* = u_{n+1}$$

and so our procedure gives us a solution of the L–L equation which automatically satisfies the required reality condition.

To impose (26), let us, first of all, consider the equation for b_{2n} (see (16)):

$$\Big(\frac{b'_{2n} - \sqrt{P}a_{2n}}{b_{2n}}\Big)' = \frac{b_{2n-1}b_{2n+1}}{b^2_{2n}} = 0$$

or in other words

$$\frac{a_{2n}}{b_{2n}} = (\ln b_{2n}f)',$$

where $f' = \dfrac{c}{\sqrt{P}}f$, and c is some constant. Then the condition (26) takes the simple form

$$Y = (b_{2n}f)^*. \quad (27)$$

However, using (27) it is possible to determine and solve the conditions that the parameters of the initial solution have to satisfy. Let us look first at the simplest case; namely, $n = 1$. The restrictions on the values of λ are contained in (23). With respect to λ_1, (23) is a cubic equation with two obvious roots: $\lambda_1 = \lambda_2, \quad \lambda_1 = \lambda_3$ (which are not interesting because they imply $a_3 = 0$ in this case). After rewriting (23) in an explicit form, we immediately determine the third root as

$$\lambda_1 = \frac{g_1g_2(\lambda_2-\lambda_3)^2}{\lambda_2\lambda_3\Big(\dfrac{\sqrt{R_2}}{\lambda_2} - \dfrac{\sqrt{R_3}}{\lambda_3}\Big)^2}. \quad (28)$$

Notice that from (23) it follows directly that

$$\frac{\lambda_2-\lambda_3}{\frac{\sqrt{R_2}}{\lambda_2}-\frac{\sqrt{R_3}}{\lambda_3}}=\frac{\lambda_1-\lambda_2}{\frac{\sqrt{R_1}}{\lambda_1}-\frac{\sqrt{R_2}}{\lambda_2}}=\frac{\lambda_1-\lambda_3}{\frac{\sqrt{R_1}}{\lambda_1}-\frac{\sqrt{R_3}}{\lambda_3}}. \tag{29}$$

Next using (18), with the help of (7) we obtain

$$b_2=\frac{1}{2}\sum_{i,j=1}^{3}\frac{1}{\sqrt{P}}\Big(\frac{\sqrt{R_i}}{\lambda_i}-\frac{\sqrt{R_j}}{\lambda_j}\Big)\det{}_2\begin{pmatrix} y_i' & y_j' \\ \lambda_i y_i & \lambda_j y_j\end{pmatrix}.$$

Thus, we see that b_2 is a linear combination of three functions.

Let us now show that each function y_{ij},

$$y_{ij}=\det{}_2\begin{pmatrix} \frac{y_i'}{\sqrt{P}} & \frac{y_j'}{\sqrt{P}} \\ \lambda_i y_i & \lambda_j y_j\end{pmatrix} f, \tag{30}$$

is a primitive solution (5) taken with an opposite sign divided by the root of the cubic function $R(\lambda)$, or in other words

$$y_{12}'=y_{12}\frac{\tilde\lambda_3 Q+\epsilon\sqrt{R(\tilde\lambda_3)P}}{\tilde\lambda_3 P+g_1g_2}, \tag{31}$$

where

$$\tilde\lambda_3=\frac{g_1g_2(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2\Big(\frac{\sqrt{R(\lambda_1)}}{\lambda_1}-\frac{\sqrt{R(\lambda_2)}}{\lambda_2}\Big)^2}$$

(where by ϵ we denote the fact of the change of the sign). From the definition (30) and (7), we can convince ourselves that (31) is satisfied. So if

$$Y=\sum y_k, \quad \text{then} \quad b_2 f\equiv\sum_{i\neq k,j\neq k}\tilde y_k\frac{\sqrt{R_i}-\sqrt{R_j}}{\lambda_i-\lambda_j},$$

where $\tilde y_k$ are the additional (to y_k) linearly independent solutions of (5).

From this last relation, it is not difficult to obtain restrictions (keeping in mind (25)) to be imposed on the parameters λ_s, c_s. Compare with the analogical calculations in the trigonometric case in [1].

7 Conclusions

The main results of this paper are the explicit formulae presenting the general solution of the elliptic Toda chain with fixed both ends and n-soliton solutions of the L-L equation which follow from it as a direct corollary. We have not addressed here the question of the application of our results to physical problems nor performed the comparison with similar results obtained by other methods. We hope to discuss these problems in a future publication.

Acknowledgments

One of us (ANL) thanks the International Scientific Foundation for supporting his research through the Grant N MMM 000.

References

[1] Belov N.A., Leznov A.N. and Zakrzewski W.J., On the solutions of the anisotropic Heisenberg equation, *J. Phys. A.*, 1994, V.27, 5607–5621.

[2] Belov N.A., Leznov A.N. and Zakrzewski W.J., Generalization of the Toda chain system to elliptic curve case, Preprint IHEP 94-55, 1994.

[3] Leznov A.N., On the general integrability of nonlinear systems of PDE in two-dimensional space, *Teor. Mat. Fiz.*, 1980, V.42, 343 (in Russian).

[4] Borovik A.E., Rubik V.N., Linear pseudopotentials and the conservation laws for the Landau-Lifschitz equation, describing the nonlinear dynamics of a ferromagnet with uniaxial anisotropy, *Teor. Mat. Fiz.*, 1981, V.46, 371–381 .

[5] Sklyanin E.K., On complete integrability of the Landau-Lifschitz equation, Preprint LOMI E-3-79, 1979.

[6] Landau L.D., Collected papers by L.D. Landau, Oxford, Pergamon, 1969, 101–114.

[7] Leznov A.N., Nonlinear Symmetries of Integrable Systems, *J. Sov. Lazer. Research*, 1992, V.3-4, 278–288.

Leznov A.N., Completely integrable systems, Preprint IHEP 92-112, 1992.

Leznov A.N., Shabat A.B. and Yamilov R.I., Canonical transformations generated by shifts in nonlinear lattices, *Phys. Lett. A*, 1993, V.174, 397–402.

Devchand Ch., Leznov A.N., Backlund transformation for super-symmetric self-dual theories for semisimple gauge groups and hierarchy of A_1 solutions, *Com. Math. Phys.*, 1994, V.160, 551–562.

[8] Leznov A.N., Razumov A.V., The canonical symmetry for integrable systems, *J. Math. Phys.*, 1994, V.35, 1738–1754.

Leznov A.N., Razumov A.V., Hamiltonian properties of the canonical symmetry, *J. Math. Phys.*, 1994, V.35, 4067–4087.

[9] Byrd P.F. and Friedman M.D., Handbook of elliptic integrals for engineers and physicists, Berlin, Springer, 1954.