Distributed Rotating Consensus Control of Second-order Leader-following Multi-Agent System with an Elliptical Orbit*

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Abstract—This paper addresses collective rotating consensus problem of second-order multi-agent system. We first define rotating consensus problem of leader-following multi-agent system in a circular orbit, and through the mapping relationship between ellipse and circle, extend rotating consensus problem to elliptical orbit, and propose a new protocol for second-order leader-following multi-agent system in an elliptical orbit. Then, based on the protocol for rotating consensus in an elliptical orbit, we transform the original system into an equivalent system by performing a model transformation and transform the rotating consensus problem to a stability problem. Based on the stable theory of Hurwitz, we verify the stability of this system and derive sufficient conditions that guarantee all leader-following multi-agents achieve rotating consensus in an elliptical orbit. Finally, the simulation results are provided to show the effectiveness of the obtained theoretical results.

Keywords—second-order multi-agent system; model transformation; elliptical orbit; rotating consensus

I. INTRODUCTION

Due to the technological advances in communication and computation, distributed coordination of networks of dynamic agents has attracted several researchers in recent years in many fields such as biology, physics and robotics and many results have been obtained in [1-7]. Li [1] et al. has dealt with $L_2 - L_\infty$ control for leader-following coordination. Hong [2] et al. was concerned with a leader-following problem for a multi-agent system with a switching interconnection topology. In [3], Zhang et al. investigated the leader-following problems of second-order multi-agent systems without relative velocity measurement. These studies [1-3], mainly through the introduction of the leader, proposed a decentralized control strategy, and enabled the leader-following multi-agent systems achieve consensus.

Recently, we have witnessed a growing interest in the study of the formation control of multi-agent systems due to the development of communication and computer. In [4], Sepulchre et al. proposed a design methodology to stabilize isolated relative equilibria in a model of identical, steered particles moving in the plane at unit speed. In [5], Pavone et al. studied this kind of collective

motions and proposed a cyclic pursuit policy. Based on the work of [4], Lin [7] investigated collective rotating motions of second-order multi-agent system.

In this paper, motivated by the work of [7], a new feedback dynamic protocol was introduced and derived sufficient conditions that the leader-following multi-agent system reach rotating consensus in an elliptical orbit.

II. GRAPH THEORY

The communication network is modeled through an undirected weighted graph $G = (S, e, \mathcal{A})$ of order n with a set of nodes $S = (s_1, s_2, \dots, s_n)$, and a set $e \subseteq S \times S$ of edges, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{n \times n}$ with non-negative elements a_{ij} for all $i, j \in I = \{1, 2, \dots, n\}$. An edge of G is denoted by $e_{ij} = (s_i, s_j)$, where the first element s_i of e_{ij} is said to be the tail of the edge and the other, s_j to be the head. Then the set of neighbours of node s_i is denoted by $N_i = \{s_j \in S : a_{ij} > 0\}$. The adjacency elements associated with the edges are positive, that is, $e_{ij} \in e \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ and $a_{ij} = a_{ji}$ for all $i, j \in I$. A diagonal matrix $\mathcal{A} = \text{diag}[\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n]$ is a degree matrix of G , whose

diagonal elements $\mathcal{A}_i = \sum_{j=1}^n a_{ij}$ for all $i = 1, 2, \dots, n$. Then,

the Laplacian of the undirected graph G is defined as $\mathbf{L} = \mathcal{A} - \mathbf{A}$. An undirected path is a sequence of ordered edges of the form $(s_i, s_{k1}), (s_{k1}, s_{k2}), \dots, (s_{kl}, s_j)$ where $s_{ij} \in S$ in an undirected graph. An undirected graph is called connected if there is a path from every node to every other node.

In this paper, we consider another graph $\tilde{G}$, associated with the system consisting of n following agents and one leader (labeled 0). Similarly, we define a diagonal matrix $\mathbf{D} \in R^{n \times n}$ to be a leader adjacency matrix associated with $\tilde{G}$ with diagonal elements d_i ,

where $d_i = a_{i0}$ for some constant $a_{i0} > 0$ if node 0 is a neighbor of node i , otherwise, $a_{i0} = 0$.

III. PROBLEM DESCRIPTION

In nature, there exist numbers collective motions such as the motion of celestial bodies and flocks of birds flying around a closed-loop system and schools of fish swimming along an approximately circular orbit, which can find important potential applications in formation flight of satellites around the earth, spacecraft docking, circular mobile sensor networks, and so on.

Due to the limited capacity of detection and amount of agents in the group, each agent can only use its neighbors information. In this paper, we considered second-order leader-following multi-agent system with the graph $\tilde{G}$. Suppose the dynamics of the i th ($i \in I$) agent is:

$$\dot{\mathbf{r}}_i = \mathbf{v}_i, \quad \dot{\mathbf{v}}_i = \mathbf{u}_i, \quad i = 1, 2, \dots, n \quad (1)$$

where the vectors $\mathbf{r}_i, \mathbf{v}_i \in C \approx R^2$ denote the position and velocity of the agent S_i and $\mathbf{u}_i(t) \in C$ is the control input. Some followers in $\tilde{G}$ are impressed by the leader and other neighbour agents, while the motion of the leader is independent.

The dynamics of the leader is described as follows:

$$\dot{\mathbf{r}}_0 = \mathbf{v}_0 \quad (2)$$

where $\mathbf{r}_0$ is the position and $\mathbf{v}_0$ is the desired constant velocity.

In this paper, our main objective was to design rules to make all leader-following multi-agents achieve rotating consensus in an elliptical orbit. Firstly, we assumed each agent moves in anti-clockwise direction. Before giving the main results, we needed to make the following Definition 1 in a circular orbit.

Definition 1. The leader-following multi-agent system (1 and 2) achieve rotating consensus in a circular orbit if

$$\lim_{t \rightarrow \infty} [\mathbf{r}_{ei}(t) - \mathbf{r}_{e0}(t)] = 0 \quad (3a)$$

$$\lim_{t \rightarrow \infty} [\mathbf{r}_{ei}(t) - \mathbf{r}_{ek}(t)] = 0 \quad (3b)$$

$$\lim_{t \rightarrow \infty} [\mathbf{v}_{ei}(t) - \mathbf{v}_{e0}(t)] = 0 \quad (3c)$$

$$\lim_{t \rightarrow \infty} [\mathbf{v}_{ei}(t) - \mathbf{v}_{ek}(t)] = 0 \quad (3d)$$

$$\lim_{t \rightarrow \infty} [\dot{\mathbf{v}}_{e01}(t) + \mathbf{j}\dot{\mathbf{v}}_{e02}(t) - \mathbf{j}\omega(\mathbf{v}_{e01}(t) + \mathbf{j}\mathbf{v}_{e02}(t))] = 0 \quad (3e)$$

$$\lim_{t \rightarrow \infty} [\dot{\mathbf{v}}_{ei1}(t) + \mathbf{j}\dot{\mathbf{v}}_{ei2}(t) - \mathbf{j}\omega(\mathbf{v}_{ei1}(t) + \mathbf{j}\mathbf{v}_{ei2}(t))] = 0 \quad (3f)$$

for all $i, k \in I$, where $\mathbf{r}_{ei} = \mathbf{r}_{ei1} + \mathbf{j}\mathbf{r}_{ei2}$,

$$\mathbf{v}_{ei} = \mathbf{v}_{ei1} + \mathbf{j}\mathbf{v}_{ei2}, \quad \mathbf{r}_{e0} = \mathbf{r}_{e01} + \mathbf{j}\mathbf{r}_{e02}, \quad \mathbf{v}_{e0} = \mathbf{v}_{e01} + \mathbf{j}\mathbf{v}_{e02}.$$

Fig. 1 shows the vector relation with one agent (labeled P) in a circular orbit, while the condition (3e) and (3f) guarantee that each agent surround a common point with a constant angular ω , where $\mathbf{j}\mathbf{v}_i$ can be regarded as the vector obtained after a rotation of the vector $\mathbf{v}_i$ through 90° in the anti-clockwise direction. Accordingly, the circle center and radius of the circle are $\mathbf{r}_i + \mathbf{j}\mathbf{v}_i\omega^{-1}$ and $\|\mathbf{v}_i\omega^{-1}\|$.

Figure 1. Vector relation in a circular orbit.

Noted that there exist a mapping relation between circle and ellipse as (4a) and (4b) if we denote

$$\mathbf{r}_{ei2} = \frac{\mathbf{r}_{i2} - d}{b}, \quad \mathbf{r}_{ei1} = \frac{\mathbf{r}_{i1} - d}{b} \quad \text{and obtain} \quad \mathbf{v}_{ei1} = \frac{\mathbf{v}_{i1}}{a},$$

$$\mathbf{v}_{ei2} = \frac{\mathbf{v}_{i2}}{b} \quad \text{by simply calculation, where} \quad \mathbf{r}_i = \mathbf{r}_{i1} + \mathbf{j}\mathbf{r}_{i2};$$

$$\mathbf{v}_i = \mathbf{v}_{i1} + \mathbf{j}\mathbf{v}_{i2}, \quad \mathbf{r}_0 = \mathbf{r}_{01} + \mathbf{j}\mathbf{r}_{02}, \quad \mathbf{v}_0 = \mathbf{v}_{01} + \mathbf{j}\mathbf{v}_{02}.$$

$$\mathbf{r}_{ei} = \frac{\mathbf{r}_{i1} - c}{a} + \mathbf{j} \frac{\mathbf{r}_{i2} - d}{b} \quad (4a)$$

$$\mathbf{v}_{ei} = \frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \quad (4b)$$

So we can get Definition 2 according to the mapping relation.

Definition 2. The leader-following multi-agent system (1 and 2) achieve rotating consensus in an elliptical orbit if

$$\lim_{t \rightarrow \infty} [\mathbf{r}_i(t) - \mathbf{r}_\theta(t)] = 0 \quad (5a)$$

$$\lim_{t \rightarrow \infty} [\mathbf{r}_i(t) - \mathbf{r}_k(t)] = 0 \quad (5b)$$

$$\lim_{t \rightarrow \infty} [\mathbf{v}_i(t) - \mathbf{v}_\theta(t)] = 0 \quad (5c)$$

$$\lim_{t \rightarrow \infty} [\mathbf{v}_i(t) - \mathbf{v}_k(t)] = 0 \quad (5d)$$

$$\lim_{t \rightarrow \infty} \left[\frac{\dot{\mathbf{v}}_{\theta 1}(t)}{a} + \mathbf{j} \frac{\dot{\mathbf{v}}_{\theta 2}(t)}{b} - \mathbf{j} \omega \left(\frac{\mathbf{v}_{\theta 1}(t)}{a} + \mathbf{j} \frac{\mathbf{v}_{\theta 2}(t)}{b} \right) \right] = 0 \quad (5e)$$

$$\lim_{t \rightarrow \infty} \left[\frac{\dot{\mathbf{v}}_{i1}(t)}{a} + \mathbf{j} \frac{\dot{\mathbf{v}}_{i2}(t)}{b} - \mathbf{j} \omega \left(\frac{\mathbf{v}_{i1}(t)}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}(t)}{b} \right) \right] = 0 \quad (5f)$$

for all $i, k \in I$.

The position and velocity of all following agents achieve consensus with the leader and surround a common point in an elliptical orbit. Then, this paper will only discuss the rotating consensus of Definition 2.

IV. ROTATING CONSENSUS

In this section, we studied rotating consensus problems of the multi-agent system (1 and 2). For simplicity of the following analysis, we assumed $\omega = 1$ and used the following consensus algorithm.

$$\mathbf{u}_i = \mathbf{u}_{i1} + \mathbf{u}_{i2} \quad (6)$$

where $\mathbf{u}_{i1} = \mathbf{j} \left(\frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \right)$ and

$$\begin{aligned} \mathbf{u}_{i2} = & -k_1 \left\{ \sum_{s_k \in N_i} a_{ij} \left[\left(\frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \right) - \left(\frac{\mathbf{v}_{k1}}{a} + \mathbf{j} \frac{\mathbf{v}_{k2}}{b} \right) \right] \right. \\ & + a_{i0} \left[\left(\frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \right) - \left(\frac{\mathbf{v}_{\theta 1}}{a} + \mathbf{j} \frac{\mathbf{v}_{\theta 2}}{b} \right) \right] \left. \right\} \\ & - k_2 \left\{ \sum_{s_k \in N_i} a_{ij} \left\{ \left[\left(\frac{\mathbf{r}_{i1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{i2}-d}{b} \right) + \mathbf{j} \left(\frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \right) \right] \right. \right. \\ & - \left[\left(\frac{\mathbf{r}_{k1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{k2}-d}{b} \right) + \mathbf{j} \left(\frac{\mathbf{v}_{k1}}{a} + \mathbf{j} \frac{\mathbf{v}_{k2}}{b} \right) \right] \left. \right\} \\ & + a_{i0} \left\{ \left[\left(\frac{\mathbf{r}_{i1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{i2}-d}{b} \right) + \mathbf{j} \left(\frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \right) \right] \right. \\ & - \left. \left[\left(\frac{\mathbf{r}_{\theta 1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{\theta 2}-d}{b} \right) + \mathbf{j} \left(\frac{\mathbf{v}_{\theta 1}}{a} + \mathbf{j} \frac{\mathbf{v}_{\theta 2}}{b} \right) \right] \right\} \end{aligned}$$

Denote

$$\begin{aligned} \delta = & [\xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_n, \eta_n]^T = [\xi_{i1} + \mathbf{j} \xi_{i2}, \eta_{i1} + \mathbf{j} \eta_{i2}, \dots, \\ & \xi_{n1} + \mathbf{j} \xi_{n2}, \eta_{n1} + \mathbf{j} \eta_{n2}]^T, \quad A = \begin{bmatrix} \mathbf{j} & -\mathbf{j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \\ B = & \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{k}_1 & \mathbf{k}_1 + \mathbf{j} \mathbf{k}_2 \end{bmatrix}, \forall i \in I \text{ and } k_1, k_2 \in R^{n \times n} > 0, \end{aligned}$$

where ξ_i and η_i are respectively

$$\xi_i = \left(\frac{\mathbf{r}_{i1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{i2}-d}{b} \right) - \left(\frac{\mathbf{r}_{\theta 1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{\theta 2}-d}{b} \right) \quad \text{and}$$

$$\eta_i = \left[\left(\frac{\mathbf{r}_{i1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{i2}-d}{b} \right) + \mathbf{j} \left(\frac{\mathbf{v}_{i1}}{a} + \mathbf{j} \frac{\mathbf{v}_{i2}}{b} \right) \right] - \left[\left(\frac{\mathbf{r}_{\theta 1}-c}{a} + \mathbf{j} \frac{\mathbf{r}_{\theta 2}-d}{b} \right) + \mathbf{j} \left(\frac{\mathbf{v}_{\theta 1}}{a} + \mathbf{j} \frac{\mathbf{v}_{\theta 2}}{b} \right) \right].$$

Then using the protocol (6), the closed-loop network dynamics can be written as

$$\dot{\delta} = (I_n \otimes A - \tilde{L} \otimes B) \delta \quad (7)$$

Where, $\tilde{L} = L + D$.

V. STABILITY ANALYSIS

Lemma 1: The matrix $\tilde{L} = L + D$ is positive if the graph $\tilde{G}$ is connected.

Lemma 2: The roots of the equation

$$s^2 + s(a_1 + a_2 \mathbf{j}) + a_3 = 0 \quad (8)$$

have negative real parts, where $a_1, a_2, a_3 \in R$,

$a_1 > 0, a_3 > 0$.

Proof: Denote the roots of (8) by $s = b_1 + b_2 \mathbf{j}$. Then, the real part and the imaginative part of the Eq.(8) is equal to (9) and (10)

$$b_1^2 - b_2^2 + a_1 b_1 - a_2 b_2 + a_3 = 0 \quad (9)$$

$$2b_1 b_2 \mathbf{j} + a_1 b_2 \mathbf{j} + b_1 a_2 \mathbf{j} = 0 \quad (10)$$

From (10), we can obtain $b_2 = -\frac{b_1 a_2}{2b_1 + a_1}$. Then

substituting b_2 into (9), we have

$$\begin{aligned} (2b_1 + a_1)^2 b_1^2 - a_2^2 b_1^2 + (a_1 b_1 + a_3)(2b_1 + a_1)^2 + a_2^2 b_1 (2b_1 + a_1) \\ = (2b_1 + a_1)^2 b_1^2 + (a_1 b_1 + a_3)(2b_1 + a_1)^2 + a_2^2 b_1 (b_1 + a_1) = 0 \end{aligned} \quad (11)$$

Noting that $a_1 > 0, a_3 > 0$, we have the Eq. (11) cannot hold if $b_1 \geq 0$. Therefore, the real part of each root of (8) is negative.

Theorem: Consider a network of second-order agents with fixed topology $\tilde{G}$ that is connected. The multi-agent system (1 and 2) with the protocol (6) reaches a rotating consensus.

Proof: By Lemma 1, we can regard $\tilde{L}$ as $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then there exists an orthogonal

matrix $W \in R^{n \times n}$ such that $W^T \tilde{L} W = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and we have

$$(W^T \otimes I_n)(I_n \otimes A - \tilde{L} \otimes B)(W \otimes I_n) = \text{diag}\{A - \lambda_1 B, A - \lambda_2 B, \dots, A - \lambda_n B\} \quad (12)$$

By calculating, the characteristic polynomial of each $A - \lambda_i B$ is

$$\det(sI - A + \lambda_i B) = s^2 + (k_1 \lambda_i + j k_2 \lambda_i - j)s + k_2 \lambda_i = 0, \quad i = 1, 2, \dots, n$$

Then by Lemma 2, the system $I_n \otimes A - \tilde{L} \otimes B$ is Hurwitz stable. Hence, $\lim_{t \rightarrow \infty} \delta(t) = 0$.

VI. SIMULATIONS

In this section, we will provide a numerical simulation to illustrate our theoretical results derived in the above sections. Consider a second-order multi-agent system with four agents and one leader. Suppose that the weight of each edge is 1. Fig.2 shows connected undirected graph with five nodes. Figs. 3 and 4 respectively show the position trajectories of the leader and following multi-agents and we can see the leader-following multi-agent system achieve a rotating consensus in an elliptical orbit under protocol (6).

Figure 2. Network topology.

Figure 3. Position trajectories of the leader.

Figure 4. Position trajectories of the following multi-agents.

VII. CONCLUSIONS

This paper investigated rotating consensus of second-order leader-following multi-agent system in an elliptical orbit. We define consensus problem of leader-following multi-agents and propose rotating consensus protocol. By performing a model transformation, give conditions that guarantee all leader-following multi-agents achieve rotating consensus in an elliptical orbit. Finally, Simulation results are provided to demonstrate the effectiveness of our theoretical results.

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