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On Reduction of the Euler Equations by Means of Two-Dimensional Algebras

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Abstract

A complete set of inequivalent two-dimensional subalgebras of the maximal Lie invariance algebra of the Euler equations is constructed. Using some of them, the Euler equations are reduced to systems of partial differential equations in two independent variables which are integrated in quadratures.

As well as developing approximate and numerical methods, finding exact solutions of the Euler equations (EEs) for an ideal incompressible fluid is an important problem of modern mathematical physics and hydrodynamics. There exist some ways to solve this problem. One of them is to use symmetry analysis [1, 2, 3]. We construct a complete set of inequivalent two-dimensional subalgebras of the maximal Lie invariance algebra of the EEs. Using some of them, we reduce the EEs to systems of partial differential equations in two independent variables which can be integrated in quadratures. As a result, we obtain classes of exact solutions of the EEs that contain arbitrary functions.

It is known [4], that the EEs

$$\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla} p = \vec{0}, \quad \operatorname{div} \vec{u} = 0 \tag{1}$$

are invariant under the infinite-dimensional algebra $A(E)$ generated by the following basis elements:

$$\begin{aligned} &\partial_t, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} \quad (a < b), \\ &D^t = t\partial_t - u^a\partial_{u^a} - 2p\partial_p, \quad D^x = x_a\partial_a + u^a\partial_{u^a} + 2p\partial_p, \\ &R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \\ &Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p. \end{aligned} \tag{2}$$

In the following, $\vec{u} = \{u^a(t, \vec{x})\}$ denotes the velocity of fluid, $p = p(t, \vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla} \cdot \vec{\nabla}$ is the Laplacian, $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0, t_1), \mathbf{R})$). The fluid density is set equal to unity. Summation over repeated indices is implied, and we have $a, b = 1, 2, 3$ and $i, j = 1, 2$.

The set of operators (1) determines the maximal Lie invariance algebra of the EEs [4].

A complete set of $A(E)$-inequivalent two-dimensional subalgebras of $A(E)$ is exhausted by the following classes of algebras:

0. $A^2_0 =< \partial_t,\ D^t >$.

1. $A^2_1(\kappa_1,\kappa_2,\varepsilon) =< \partial_t,\ D^x + \kappa_1 D^t + \kappa_2 J_{12} + Z(\varepsilon) >$ where $\varepsilon(1-\kappa_1) = 0$ and $\kappa_1 \neq 0$.

2. $A^2_2(\kappa,\mu) =< \partial_t,\ J_{12} + \kappa D^t + R(0,0,\mu) >$ where $\kappa \neq 0$.

3. $A^2_3(\kappa) =< \partial_t,\ R(0,0,1) + \kappa D^t >$, where $\kappa \neq 0$.

4. $A^2_4(\kappa_1,\kappa_2) =< D^t + \kappa_1 J_{12},\ D^x + \kappa_2 J_{12} >$.

5. $A^2_5(\kappa,\mu,\varepsilon) =< D^t + \kappa D^x,\ J_{12} + R(0,0,\mu|t|^\kappa) + Z(\varepsilon|t|^{2(\kappa-1)}) >$, where we can assume that $\mu \in \{-1;0;1\}$ and $\varepsilon \in \{-1;0;1\}$ if $\mu = 0$.

6. $A^2_6(\kappa_1,\kappa_2,\sigma,\mu,\nu,\varepsilon) = <D^t + \kappa_1 D^x + \kappa_2 J_{12},\ R(|t|^\sigma(\mu\cos\tau,\mu\sin\tau,\nu)) + Z(\varepsilon|t|^{\kappa_1-2+\sigma})>$, where $\tau = \kappa_2 \ln|t|$. The coefficients μ and ν satisfy either the condition $\mu^2+\nu^2 = 1$ (then we can assume that $\varepsilon(\kappa_1+\sigma-1) = 0$) or the condition $\mu = \nu = 0$ (then we can assume that $\varepsilon = 1$). In the case $\kappa_2 = 0$, we can take $\mu = 0$.

7. $A^2_7(\kappa_1,\kappa_2) =< \partial_t + \kappa_1 J_{12},\ D^x + \kappa_2 J_{12} >$.

8. $A^2_8(\kappa,\mu,\varepsilon) =< \partial_t + \kappa D^x,\ J_{12} + R(0,0,\mu e^{\kappa t}) + Z(\varepsilon e^{2\kappa t}) >$, where we can assume that $\kappa,\mu \in \{-1;0;1\}$ and $\varepsilon \in \{-1;0;1\}$ if $\mu = 0$.

9. $A^2_9(\kappa_1,\kappa_2,\sigma,\mu,\nu,\varepsilon) =< \partial_t + \kappa_1 D^x + \kappa_2 J_{12},\ R(e^{\sigma t}(\mu\cos\tau,\mu\sin\tau,\nu)) + Z(\varepsilon e^{(\kappa_1+\sigma)t}) >$, where $\tau = \kappa_2 t$. The coefficients μ and ν satisfy either the condition $\mu^2+\nu^2 = 1$ (then we can assume that $\varepsilon \in \{-1;0;1\}$ and $\varepsilon(\kappa_1+\sigma) = 0$) or the condition $\mu = \nu = 0$ (then we can assume that $\varepsilon = 1$). In the case $\kappa_2 = 0$, we can take $\mu = 0$.

10. $A^2_{10} =< D^x,\ J_{12} >$.

11. $A^2_{11}(\vec{m}) =< D^x,\ R(\vec{m}) >$, where $\vec{m} = \vec{m}(t)$ is a smooth vector-function of t, $\vec{m} \not\equiv \vec{0}$. The algebras $A^2_{11}(\vec{m})$ and $A^2_{11}(\tilde{\vec{m}})$ are equivalent if $\exists \varepsilon,\delta \in \mathbf{R}$, $\exists B \in O(3)$, $\exists C \neq 0$: $\tilde{\vec{m}}(\tilde{t}) = CB\vec{m}(t)$, where $\tilde{t} = te^{-\varepsilon} + \delta$.

12. $A^2_{12}(\kappa,\eta) =< D^x + \kappa J_{12},\ R(0,0,\eta(t)) >$, where $\eta = \eta(t)$ is a smooth function of t, $\eta \not\equiv 0$, $\kappa \neq 0$. The algebras $A^2_{12}(\kappa,\eta)$ and $A^2_{12}(\tilde{\kappa},\tilde{\eta})$ are equivalent if $\tilde{\kappa} = \kappa$ and $\exists \varepsilon,\delta \in \mathbf{R}$, $\exists C \neq 0 : \tilde{\eta}(\tilde{t}) = C\eta(t)$, where $\tilde{t} = te^{-\varepsilon} + \delta$.

13. $A^2_{-1}(\kappa,\chi) =< D^x + \kappa J_{12},\ Z(\chi(t)) >$, where $\chi = \chi(t)$ is a smooth function of t, $\chi \not\equiv 0$, $\kappa \in \mathbf{R}$. The algebras $A^2_{-1}(\kappa,\chi)$ and $A^2_{-1}(\tilde{\kappa},\tilde{\chi})$ are equivalent if $\tilde{\kappa} = \kappa$ and $\exists \varepsilon,\delta \in \mathbf{R}$, $\exists C \neq 0 : \tilde{\chi}(\tilde{t}) = C\chi(t)$, where $\tilde{t} = te^{-\varepsilon} + \delta$.

14. $A^2_{13}(\rho^1,\chi^1,\rho^2,\chi^2) =< J_{12} + R(0,0,\rho^2) + Z(\chi^2),\ R(0,0,\rho^1) + Z(\chi^1) >$, where ρ^i and χ^i are smooth functions of t, $(\rho^1,\chi^1) \not\equiv (0,0)$ and $\rho^1_{tt}\rho^2 - \rho^1\rho^2_{tt} \equiv 0$. The algebras $A^2_{13}(\rho^1,\chi^1,\rho^2,\chi^2)$ and $A^2_{13}(\tilde{\rho}^1,\tilde{\chi}^1,\tilde{\rho}^2,\tilde{\chi}^2)$ are equivalent if $\exists C_1 \neq 0$, $\exists \varepsilon_1,\varepsilon_2,\delta,C_2 \in \mathbf{R}$,

$\exists\theta \in C^\infty((t_0, t_1), \mathbf{R})$:

$$\begin{aligned}&\tilde\rho^1(\tilde t) = C_1 e^{-\varepsilon_2}\rho^1(t), \quad \tilde\rho^2(\tilde t) = e^{-\varepsilon_2}(\rho^2(t) + C_2\rho^1(t)),\\ &\tilde\chi^1(\tilde t) = C_1 e^{2(\varepsilon_1-\varepsilon_2)}(\chi^1(t) + \theta_{tt}(t)\rho^1(t) - \theta(t)\rho^1_{tt}(t)),\\ &\tilde\chi^2(\tilde t) = e^{2(\varepsilon_1-\varepsilon_2)}(\chi^2(t) + \theta_{tt}(t)\rho^2(t) - \theta(t)\rho^2_{tt}(t)+\\ &\qquad\qquad C_2(\chi^1(t) + \theta_{tt}(t)\rho^1(t) - \theta(t)\rho^1_{tt}(t))),\end{aligned} \tag{3}$$

where $\tilde t = te^{-\varepsilon_1} + \delta$.

15. $A^2_{14}(\vec m^1, \chi^1, \vec m^2, \chi^2) =< R(\vec m^1(t)) + Z(\chi^1(t)), R(\vec m^2(t)) + Z(\chi^2(t)) >$, where $\vec m^i$ and χ^i are smooth functions such that $\vec m^1_{tt}\cdot\vec m^2 - \vec m^1\cdot\vec m^2_{tt} = 0$ and $\not\exists C_i \in \mathbf{R} : C_i(\vec m^i, \chi^i) \equiv (\vec 0, 0)$. The algebras $A^2_{14}(\vec m^1, \chi^1, \vec m^2, \chi^2)$ and $A^2_{14}(\vec{\tilde m}^1, \tilde\chi^1, \vec{\tilde m}^2, \tilde\chi^2)$ are equivalent if $\exists\varepsilon_1, \varepsilon_2, \delta \in \mathbf{R}$, $\exists B \in O(3)$, $\exists\{a_{ij}\}_{i,j=1,2} : \det\{a_{ij}\} \neq 0$, $\exists\vec l \in C^\infty((t_0, t_1), \mathbf{R}^3)$:

$$\begin{aligned}&\vec{\tilde m}^i(\tilde t) = e^{-\varepsilon_2} a_{ij} B\vec m^j(t),\\ &\tilde\chi^i(\tilde t) = e^{2(\varepsilon_1-\varepsilon_2)} a_{ij}(\chi^j(t) + \vec l_{tt}(t)\cdot\vec m^j(t) - \vec l(t)\cdot\vec m^j_{tt}(t)),\end{aligned} \tag{4}$$

where $\tilde t = te^{-\varepsilon_1} + \delta$.

Using the algebras $A^2_1 - A^2_{14}$ (sometimes, when additional restrictions for parameters are satisfied), we can construct an ansatz that reduces the EEs to a system of partial differential equations in two independent variables. To reduce the EEs, we use the algebras A^2_{13} and A^2_{14}. They are different from other two-dimensional subalgebras of $A(E)$ in that the reduced equations obtained by means of them can be integrated completely in quadratures.

Let us describe how the Euler equations are reduced by means of the algebra $A^2_{13}(\rho^1, \chi^1, \rho^2, \chi^2)$. It is suitable for construction of an ansatz only for such t that $\rho^1(t) \neq 0$. If this condition is satisfied, the algebra given above is equivalent to the algebra

$$A^2_{13}(\rho, 0, \hat\rho, \chi_t), \quad \text{where} \quad \hat\rho(t) = \varepsilon\rho(t)\int(\rho(t))^{-2}dt, \quad \varepsilon \in \{0; 1\}. \tag{5}$$

An ansatz constructed by means of algebra (5) has the form:

$$\begin{aligned}u^1 &= x_1 w^3 - x_2 r^{-2}(w^1 - \chi(t)),\\ u^2 &= x_2 w^3 + x_1 r^{-2}(w^1 - \chi(t)),\\ u^3 &= (\rho(t))^{-1}(w^2 + \rho_t(t)x_3 + \varepsilon\arctan x_2/x_1),\\ p &= s - \tfrac{1}{2}\rho_{tt}(t)(\rho(t))^{-1}x_3^2 + \chi_t(t)\arctan x_2/x_1,\end{aligned} \tag{6}$$

where $w^a = w^a(z_1, z_2)$ and $q = q(z_1, z_2)$ are new unknown functions, $z_1 = t$, $z_2 = r = (x_1^2 + x_2^2)^{1/2}$. Substituting ansatz (6) into the EEs, we obtain the system of differential equations (for the functions w^a and s):

$$\begin{aligned}&w^3_1 + z_2 w^3 w^3_2 + (w^3)^2 - z_2^{-4}(w^1 - \chi)^2 + z_2^{-1}s_2 = 0,\\ &w^1_1 + z_2 w^3 w^1_2 = 0,\\ &w^2_1 + z_2 w^3 w^2_2 + \varepsilon z_2^{-2}(w^1 - \chi) = 0,\\ &2w^3 + z_2 w^3_2 + \rho_1\rho^{-1} = 0.\end{aligned} \tag{7}$$

Equations (7) imply that

$$w^3 = -\tfrac{1}{2}\rho_t\rho^{-1} + \eta z_2^{-2}, \tag{8}$$

where $\eta = \eta(t)$ is an arbitrary smooth function of $z_1 = t$,

$$w^1 = \varphi^1(z), \quad w^2 = \varphi^2(z) - \frac{\varepsilon}{2}\int \frac{\varphi^1(z) - \chi(\tau)}{z + \int \rho(\tau)\eta(\tau)d\tau}\,\rho(\tau)\,d\tau, \tag{9}$$

where $\tau = t$, $z = \frac{1}{2}\rho(t)r^2 - \int \rho(t)\eta(t)dt$, and

$$q = \frac{1}{4}\left(\left(\frac{\rho_t}{\rho}\right)_t - \frac{1}{2}\left(\frac{\rho_t}{\rho}\right)^2\right)r^2 + \frac{\rho}{4}\int \frac{(\varphi^1(z) - \chi(\tau))^2}{(z + \int \rho(\tau)\eta(\tau)d\tau)^2}\,dz - \eta_t \ln|r| - \frac{\eta^2}{2r^2}. \tag{10}$$

In some cases, the expression for w^2 is simplified:

a) $w^2 = \varphi^2(z)$ if $\varepsilon = 0$;

b) $w^2 = \varphi^2(z) - \frac{1}{2}z^{-1}\varphi^1(z)\int \rho(t)dt + \frac{1}{2}z^{-1}\int \chi(t)\rho(t)dt$ if $\varepsilon = 0$ and $\eta \equiv 0$.

Ansatz (6) and formulae (8)–(10) determine a class of solutions of the EEs.

Let us describe how the Euler equations are reduced by means of the algebra $A^2_{14}(\vec m^1, \chi^1, \vec m^2, \chi^2)$. An ansatz corresponding to this algebra can be obtained only for such t that $\operatorname{rank}(\vec m^1(t), \vec m^2(t)) = 2$. For these values of t, the parameter-function $\chi^i = 0$ can be made to vanish by means of the equivalence transformation (4). An ansatz corresponding to the algebra $A^2_{14}(\vec m^1, 0, \vec m^2, 0)$ has the form

$$\begin{aligned}\vec u &= \vec w + \lambda^{-1}(\vec n^i\cdot\vec x)\vec m^i_t - \lambda^{-1}(\vec k\cdot\vec x)\vec k_t,\\ p &= s - \tfrac{1}{2}\lambda^{-1}(\vec m^i_{tt}\cdot\vec x)(\vec n^i\cdot\vec x) - \tfrac{1}{2}\lambda^{-2}(m^i_{tt}\cdot\vec k)(\vec n^i\cdot\vec x)(\vec k\cdot\vec x),\end{aligned} \tag{11}$$

where $\vec w = (w^1, w^2, w^3)$, $w^a = w^a(z_1, z_2)$ and $q = q(z_1, z_2)$ are new unknown functions, $z_1 = t$, $z_2 = \vec k\cdot\vec x$, $\vec m^1_{tt}\cdot\vec m^2 - \vec m^1\cdot\vec m^2_{tt} = 0$, $\vec k = \vec m^1\times\vec m^2$, $\vec n^1 = \vec m^2\times\vec k$, $\vec n^2 = \vec k\times\vec m^1$, and $\lambda = \lambda(t) = \vec k\cdot\vec k \neq 0$.

Substituting ansatz (11) into the EEs, we obtain the system of differential equations for the functions w^a and q:

$$\vec w_1 + (\vec k\cdot\vec w)(\vec w_2 - \lambda^{-1}\vec k_t) + \lambda^{-1}(\vec n^i\cdot\vec w)\vec m^i_t + s_2\vec k + z_2\vec e = \vec 0, \tag{12}$$

$$\vec k\cdot\vec w_2 = 0, \tag{13}$$

where $z_1 = t$ and $\vec e = \vec e(t) = 2\lambda^{-2}(\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t)\vec k_t\times\vec k + \lambda^{-2}(2\vec k_t\cdot\vec k_t - \vec k_{tt}\cdot\vec k)\vec k$.

Equation (13) is integrated with respect to z_2 to the following expression: $\vec k\cdot\vec w = \psi(t)$. Here $\psi = \psi(t)$ is an arbitrary smooth function of $z_1 = t$, which can be made to vanish by means of the transformation generated by the operator $R(\vec l)$, where the vector-function $\vec l$ is a solution of the system

$$\vec l_{tt}\cdot\vec m^i - \vec l\cdot\vec m^i_{tt} = 0, \quad \vec k\cdot(\vec l_t - \lambda^{-1}(\vec n^i\cdot\vec l)m^i_t + \lambda^{-1}(\vec k\cdot\vec l)\vec k_t) + \eta = 0.$$

Therefore, without loss of generality we may assume that $\vec k\cdot\vec w = 0$.

Let $f^i = f^i(z_1, z_2) = \vec m^i\cdot\vec w$. Since $\vec m^1_{tt}\cdot\vec m^2 = \vec m^1\cdot\vec m^2_{tt}$, it follows that

$$\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = C = \text{const},$$

and we may assume that $C\in\{0;1\}$.

Let us multiply the scalar equation (12) by $\vec{m}^i$ and $\vec{k}$. We then obtain the linear system of PDEs with variable coefficients in the functions f^i and s:

$$f^i_1 + C\lambda^{-1}((\vec{m}^i\cdot\vec{m}^2)f^1 - (\vec{m}^i\cdot\vec{m}^1)f^2) - 2C\lambda^{-2}((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i)z_2 = 0, \qquad (14)$$
$$s_2 = 2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)f^i + \lambda^{-2}(\vec{k}_{tt}\cdot\vec{k} - 2\vec{k}_t\cdot\vec{k}_t)z_2.$$

Let us consider two possible cases.

A. Let $C=0$

Then equations (14) have the forms $f^i_1=0$, i.e. $f^i=f^i(z)$, where $z:=z_2$. Therefore, in this case we obtain the following solution of the EEs:

$$\vec{u} = \lambda^{-1}(f^i(z) + \vec{m}^i_t\cdot\vec{x})\vec{n}^i - \lambda^{-1}(\vec{k}_t\cdot\vec{x})k,$$
$$p = 2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)\int f^i(z)dz + \tfrac{1}{2}\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k} - 2\vec{k}_t\cdot\vec{k}_t)z^2 - \qquad (15)$$
$$\tfrac{1}{2}\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x}) - \tfrac{1}{2}\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}),$$

where $\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = 0$, $\vec{k}=\vec{m}^1\times\vec{m}^2$, $\vec{n}^1=\vec{m}^2\times\vec{k}$, $\vec{n}^2=\vec{k}\times\vec{m}^1$, $\lambda=|\vec{k}|^2$, and $z=\vec{k}\cdot\vec{x}$.

Note. The equation

$$\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = 0 \qquad (16)$$

can be easily solved in the following way: Let us choose arbitrary smooth vector-functions $\vec{m}^1$ and $\vec{l}$ such that $\vec{m}^1(t)\neq\vec{0}$, $\vec{l}(t)\neq\vec{0}$, and $\vec{m}^1(t)\cdot\vec{l}(t)=0$ for the values of t considered. Then the vector-function $\vec{m}^2=\vec{m}^2(t)$ is taken in the form

$$\vec{m}^2(t) = \rho(t)\vec{m}^1 + \vec{l}(t). \qquad (17)$$

Equation (16) implies $\rho(t)=\int(\vec{m}^1\cdot\vec{m}^1)^{-1}(\vec{m}^1_t\cdot\vec{l} - \vec{m}^1\cdot\vec{l}_t)dt$.

B. Let $C=1$

In this case, the general solution of system (14) can be presented in the form

$$f^i(z_1,z_2) = \theta^{ij}(t)g^j(z) + \theta^{i0}(t)z,$$

where $g^i=g^i(z)$ are arbitrary smooth functions of $z:=z_2$, $(\theta^{1i}(t),\theta^{2i}(t))$ are linearly independent solutions of the system

$$\theta^i_t + \lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1 - \lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2 = 0, \qquad (18)$$

and $(\theta^{10}(t),\theta^{20}(t))$ is a particular solution of the inhomogeneous system

$$\theta^i_t + \lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1 - \lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2 = 2\lambda^{-2}((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i). \qquad (19)$$

Substituting the expressions for f^i in ansatz (11), we obtain the following solution of the Euler equations:

$$\vec{u} = \lambda^{-1}(\theta^{ij}g^j(z) + \theta^{i0}z)\vec{n}^i + \lambda^{-1}(\vec{n}^i\cdot\vec{x})\vec{m}^i_t - \lambda^{-1}z\vec{k}_t,$$
$$p = 2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)(\theta^{ij}\int g^j(z)dz + \tfrac{1}{2}\theta^{i0}z^2) + \tfrac{1}{2}\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k} - 2\vec{k}_t\cdot\vec{k}_t)z^2 - \qquad (20)$$
$$\tfrac{1}{2}\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x}) - \tfrac{1}{2}\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}).$$

Here $\vec{m}^1_t \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_t = 1$, $\vec{k} = \vec{m}^1 \times \vec{m}^2$, $\vec{n}^1 = \vec{m}^2 \times \vec{k}$, $\vec{n}^2 = \vec{k} \times \vec{m}^1$, $\lambda = |\vec{k}|^2$, and $z = \vec{k} \cdot \vec{x}$.

Note. A solution of the equation

$$\vec{m}^1_t \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_t = 1 \qquad (21)$$

can also be found in form (17). Equation (21) implies that

$$\rho(t) = \int |\vec{m}^1|^{-2}(\vec{m}^1_t \cdot \vec{l} - \vec{m}^1 \cdot \vec{l}_t - 1)dt.$$

We notice that results given in [5] partly coincide with ours. But it seems to us that the paper [5] includes small incorrectnesses.

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