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$*$-Representations of the Quantum Algebra $U_q(sl(3))$

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Abstract

Studied in this paper are real forms of the quantum algebra $U_q(sl(3))$. Integrable operator representations of $*$-algebras are defined. Irreducible representations are classified up to a unitary equivalence.

1 Introduction

There is a quantum analog of the enveloping algebra $U_q(\mathbf{J})$, where $q \in \mathbf{C} \setminus \{0, \pm 1\}$ is a parameter (see [1]) associated with each complex simple Lie algebra $\mathbf{J}$. The quantum algebra $U_q(sl(3))$ is a $\mathbf{C}$-algebra generated by $k_i^{\pm 1}$, X_i, Y_i, $i = 1, 2$, satisfying the relations:

$$[k_1, k_2] = 0, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i X_j = q^{a_{ij}} X_j k_i, \qquad k_i Y_j = q^{-a_{ij}} Y_j k_i, \tag{1}$$

$$[X_i, Y_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, \tag{2}$$

$$\begin{gathered} X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 = 0, \qquad i \neq j, \\ Y_i^2 Y_j - (q + q^{-1}) Y_i Y_j Y_i + Y_j Y_i^2 = 0, \qquad i \neq j, \end{gathered} \tag{3}$$

where

$$a_{ij} = \begin{cases} -1/2, & i \neq j \\ 1, & i = j \end{cases} \qquad \text{and} \qquad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

It is natural for such algebras to study representations of their real forms. Some representations of $U_q(sl(3))$ were studied by different authors. In particular, a $*$-representation of $U_q(sl(2))$ was studied in [7]. All finite-dimensional representations of $U_q(sl(N))$, which are equivalent to a representation of the real form $su_q(N)$ (defined by the involution $X_i^* = Y_i$, $k_i^* = k_i$, for $q \in \mathbf{R}$ and $k_i^* = k_i^{-1}$, for $q \in \mathbf{T}$) were investigated in [1]. The paper [6] studied the so-called Harish-Chandra modules of $su_q(N, 1)$, $q \in \mathbf{R}$, i. e., such representations that the spectra of k_i belong to $q^{\mathbf{Z}/2}$, besides that, restriction of the representations to the subalgebra $su_q(N)$ is decomposed into an orthogonal sum of irreducible representations of $su_q(N)$ in such a way that each of them is contained in the decomposition once (a quantum anlog of the representations of $su(N, 1)$ which are integrable to the group $SL(N, \mathbf{R})$). Representations of another real form, the $*$-algebra $sl_q(3, \mathbf{R})$, $q \in \mathbf{R}$, defined by the involution $k_i^* = k_i^{-1}$, $X_i^* = X_i$, $Y_i^* = Y_i$, is described in [10].

In this paper we study representations of real forms of $U_q(sl(3))$ by using a technique of semilinear relations developed in [9], [5]. Since the use of unbounded operators is necessary in each case, we give definitions of operator representations of $*$-algebras in a Hilbert space H. In accordance with these definitions, we describe all irreducible representation up to a unitary equivalence.

2 Object

The quantum algebra $U_q(sl(3))$ is the $\mathbf{C}$-algebra generated by k_i, k_i^{-1}, X_i , Y_i, $i = 1, 2$, satisfying the relations:

$$\begin{aligned} &[k_1, k_2] = 0, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1, \\ &k_i X_j = q^{a_{ij}} X_j k_i, \qquad k_i Y_j = q^{-a_{ij}} Y_j k_i, \end{aligned} \tag{4}$$

$$[X_i, Y_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, \tag{5}$$

$$\begin{aligned} X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 = 0, i \neq j, \\ Y_i^2 Y_j - (q + q^{-1}) Y_i Y_j Y_i + Y_j Y_i^2 = 0, i \neq j, \end{aligned} \tag{6}$$

where

$$a_{ij} = \begin{cases} -1/2, & i \neq j \\ 1, & i = j \end{cases} \quad \text{and} \quad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Remark 1 Transposition $k_i \leftrightarrow k_i^{-1}$ gives $U_q(sl(3)) \leftrightarrow U_{q^{-1}}(sl(3))$.

3 Real Forms ($*$-algebras)

Consider real forms of $U_q(sl(3))$. We will assume by definition that the real form of an algebra A is determined by such an involution that:

1) it transforms a generator into a linear combination of generators,

2) axiom $(AB)^* = B^* A^*$ does not lead to a relation which is not a corollary of (1)–(3).

Consider nonisomorphic $*$-algebras, which are real forms of $U_q(sl(3))$. Set $t = q^{1/2}$.

Proposition 1 *There are six real forms of* $U_q(sl(3))$

A_1: $k_i^* = k_i$, $X_1^* = Y_i$, $t \in \mathbf{R}$; $k_i^* = k_i^{-1}$, $q \in \mathbf{T}$;
A_2: $k_i^* = k_i$, $X_1^* = -Y_1$, $X_2^* = -Y_2$, $t \in \mathbf{R}$; $k_i^* = k_i^{-1}$, $q \in \mathbf{T}$;
A_3: $k_i^* = k_i$, $X_i^* = -Y_i$, $t \in \mathbf{R}$; $k_i^* = k_i^{-1}$, $q \in \mathbf{T}$;
A_4: $k_i^* = k_i^{-1}$, $X_i^* = X_i$, $Y_i^* = Y_i$, $t \in \mathbf{R}$; $k_i^* = k_i$, $q \in \mathbf{T}$;
A_5: $k_i^* = k_j^{-1}$, $X_i^* = X_j$, $Y_i^* = Y_j$, $i \neq j$, $t \in \mathbf{R}$; $k_i^* = k_j$, $q \in \mathbf{T}$, $i \neq j$;
A_6: $k_i^* = k_j$, $X_i^* = Y_j$, $i \neq j$, $t \in \mathbf{R}$; $k_i^* = k_j^{-1}$, $i \neq j$, $q \in \mathbf{T}$;

where $\mathbf{T} = \{\mathbf{z} \in \mathbf{C} \mid |\mathbf{z}| = \mathbf{1}\}$.

For $q \in T$ $*$-algebras A_1, A_2, A_3 are isomorphic.

Remark 2 1) There are no $*$-structure for $U_q(sl(3))$ when $t \notin \mathbf{R} \cup \mathbf{T}$.

2) Only $*$-algebras A_1, A_2, A_3, A_5 with $t \in \mathbf{R}$; A_4, A_6 with $t \in \mathbf{T}$ are $*$-Hopf-algebras (which means that involution agrees with comultiplication, counit and antipode). A complete description of $*$-Hopf algebras of the quantum algebras $U_q(\mathbf{J})$ ($\mathbf{J}$ a is simple Lie algebra) is given, in particular, in [8].

4 $*$-Representations of $U_q(sl(3))$

To study the representations of different real forms of $U_q(sl(3))$ with using unbounded operators is necessary. Following [5], we give

Definition 1 *A collection of operators k_i, X_i, Y_i is called a representation of A_i, $i = \overline{1,3}$ in a Hilbert space H if there exists a dense set $\Phi \subset H$ such that:*

a) Φ is invariant with respect to k_i, X_i, Y_i, $E(\Delta)$, $\Delta \in \mathbf{B}(\mathbf{R}^3)$, where $E(\cdot)$ is a joint resolution of indentity for the family of commuting selfadjoint operators (k_i, $X_1^* X_1$, $i = 1, 2$);

b) Φ consists of bounded vectors for k_i, $X_1^* X_1$, $X_2^* X_2$ $i = 1, 2$;

c) relations (1)–(3) hold on Φ.

Under such a definition the technique of semilinear relation developed in [5] allows one to describe all the irreducible representations of the $*$-algebras up to a unitary equivalence. A detailed study of the representations of $*$-algebras A_1, A_2 is given in [5]. In particular, it was proved that there are representations of A_2 such that the spectrum of operator k_i, $i = 1, 2$ does not belong to $q^{\mathbf{Z}/2}$ under such a definition. The following theorem gives a full description of the irreducible representations of $*$-algebra A_3.

Theorem 1 *For $q \in R$, $q > 1$, the $*$-algebra A_1 has the following irreducible representations:*

a)

$$
\begin{aligned}
k_1 f_{m_1,m_2,m_3} &= q^{m_1+(\beta+1+m_2-m_3)/2} f_{m_1,m_2,m_3}, \\
k_2 f_{m_1,m_2,m_3} &= q^{m_3-m_2-(m_1+\delta+1)/2} f_{m_1,m_2,m_3}, \\
X_1 f_{m_1,m_2,m_3} &= \sqrt{[m_1 - m_3 + 1]_q [\beta + m_1 + m_2 + 1]_q} f_{m_1+1,m_2,m_3}, \\
X_2 f_{m_1,m_2,m_3} &= \sqrt{[m_2]_q [\delta + m_2]_q} \left(\prod_{r=0}^{m_1-1} \frac{[\beta + m_2 + 1]_q}{[\beta + m_2 + r]_q} \right)^{1/2} \times \\
&\quad \left(\prod_{r=0}^{m_3-1} \frac{[\beta + m_2 + r - 1]_q}{[\beta + m_2 + r + 1]_q} \right)^{1/2} f_{m_1,m_2-1,m_3} + \\
&\quad g(m_1, m_2, m_3) \left(\prod_{r=0}^{m_2-1} \frac{[\beta + m_2 + m_3 - r - 1]_q}{[\beta + m_2 + m_3 - r + 1]_q} \right)^{1/2} f_{m_1,m_2,m_3+1},
\end{aligned}
$$

where

$$g(m_1,m_2,m_3)=\begin{cases}\sqrt{\dfrac{[m_3+1]_q[\delta-m_3-\beta]_q[m_1-m_3]_q}{[\beta+m_3+1]_q}}\times \\ \left(\prod_{r=0}^{m_1-m_3-2}\dfrac{[2+r]_q}{[1+r]_q}\right)^{1/2}, & \text{if } m_3<m_1+1,\\ 0, & \text{if } m_3=m_1,\\ \sqrt{\dfrac{[m_3+1]_q[\delta-m_3-\beta]_q[m_1-m_3]_q}{[\beta+m_3+1]_q}}, & \text{if } m_3=m_1+1,\end{cases}$$

with $0\le m_3\le m_1,\ m_2\ge 0,\ 0\le m_3\le s-1,\ \beta\ge 0,\ \delta=\beta+s-1;$

b)

$$\begin{aligned}k_1 f_{m_1,m_2,m_3} &= q^{-\beta-m_1-1+(m_3-m_2)/2} f_{m_1,m_2,m_3},\\ k_2 f_{m_1,m_2,m_3} &= q^{m_2-m_3+(m_1+\delta)/2} f_{m_1,m_2,m_3},\\ X_1 f_{m_1,m_2,m_3} &= \sqrt{[m_1-m_3]_q[\beta+m_1+m_2]_q} f_{m_1-1,m_2,m_3},\\ X_2 f_{m_1,m_2,m_3} &= \sqrt{[m_2+1]_q[\delta+m_2]_q}\left(\prod_{r=0}^{m_1-1}\frac{[\beta+m_2+r+2]_q}{[\beta+m_2+r+1]_q}\right)^{1/2}\times\\ &\left(\prod_{r=0}^{m_3-1}\frac{[\beta+m_2+m_3-r-1]_q}{[\beta+m_2+m_3-r+1]_q}\right)^{1/2} f_{m_1,m_2+1,m_3}+\\ &\sqrt{[m_3]_q[\delta-\beta-m_3]_q}\left(\prod_{r=0}^{m_1-m_3-1}\frac{[r]_q}{[r+1]_q}\right)^{1/2}\times\\ &\left(\prod_{r=0}^{m_2-1}\frac{[\beta+m_3+r-1]_q}{[\beta+m_3+r+1]_q}\right)^{1/2} f_{m_1,m_2,m_3-1},\end{aligned}$$

where $0\le m_3\le m_1,\ m_2\ge 0,\ 0\le m_3\le s,\ \beta\ge 0,\ \delta=\beta+s+1.$

c)

$$\begin{aligned}k_1 f_{m_1,m_2,m_3} &= q^{(\alpha-\beta+m_3-m_2-1)/2+m_1} f_{m_1,m_2,m_3},\\ k_2 f_{m_1,m_2,m_3} &= q^{m_2-m_3+(\delta-m_1+1)/2} f_{m_1,m_2,m_3},\\ X_1 f_{m_1,m_2,m_3} &= \sqrt{\{\alpha+m_1+m_3\}_q\{\beta-m_1+m_2\}_q} f_{m_1+1,m_2,m_3},\\ X_2 f_{m_1,m_2,m_3} &= \sqrt{[m_2+1]_q\frac{\{\delta+\alpha+m_2+1\}_q\{\beta+m_2+1\}_q}{[\alpha+\beta+m_2+1]_q}}\times\\ &\left(\prod_{s=0}^{m_1-1}\frac{\{\beta+m_2-s\}_q}{\{\beta+m_2+1-s\}_q}\right)^{1/2}\times\\ &\left(\prod_{s=0}^{m_3-1}\frac{[\alpha+\beta+m_2+s]_q}{[\alpha+\beta+m_2+s+2]_q}\right)^{1/2} f_{m_1,m_2+1,m_3}+\\ &\sqrt{[m_3]_q\frac{\{\alpha+m_3-1\}_q\{\beta-\delta+m_3-1\}}{[\alpha+\beta+m_3]_q}}\times\\ &\left(\prod_{s=0}^{m_1-1}\frac{\{\alpha+m_3+s\}_q}{\{\alpha+m_3+s-1\}_q}\right)^{1/2}\times\\ &\left(\prod_{s=0}^{m_2-1}\frac{[\alpha+\beta+m_3+s-1]_q}{[\alpha+\beta+m_3+s+1]_q}\right)^{1/2} f_{m_1,m_2,m_3-1},\end{aligned}$$

where $\alpha+\beta\geq 0$, $m_2\geq 0$, $m_3\geq 0$, $m_1\in\mathbf{Z}$.

d) one-dimensional: $X_i=Y_i=0$, $k_i=\pm 1,\pm i$.

where $[\alpha]_q=(q^\alpha-q^{-\alpha})/(q^1-q^{-1})$, $\{\alpha\}_q=(q^\alpha+q^{-\alpha})/(q^1-q^{-1})$

$*$-Representations of $*$-algebras A_4, A_5 provided that $q>1$ can be also studied by the method of semilinear relations.

Definition 2 *A collection of operators* k_i, X_i, Y_i *(*$k_i^*=k_i^{-1}$, $X_1=X_1^*$, $Y_2=Y_2^*$, X_2, Y_1 *are symmetric) is called a representation of* $sl_q(3,\mathbf{R})$, $q\in\mathbf{R}$ *in a Hilbert space* H *if there exists a dence set* $\Phi\subset H$ *such that:*

a) Φ *is invariant with respect to* k_i, X_i, Y_i, $E(\delta)$, $\delta\in\mathbf{B}(\mathbf{R}^2)$, *where* $E(\cdot)$ *is a joint resolution of indentity for the family of commuting selfadjoint operators* X_1, Y_2*;*

b) Φ *consists of bounded vectors for the operators* X_1, Y_2*;*

c) relations (1)–(3) *hold on* Φ.

Theorem 2 *For* $q\in\mathbf{R}$, $q>1$, *the* $*$*-algebra* $sl_q(3,\mathbf{R})$ *has the following irreducible representations:*

1) *a one-dimensional:* $X_i=Y_i=0$, $k_i=\pm 1,\pm i$*;*

2) *an infinite-dimensional: in* $l_2(\mathbf{Z}^2)=\{f_{k,m}\}$

$$k_1 f_{k,m}=f_{k-2,m-1},\qquad k_2 f_{k,m}=f_{k+1,m+2},$$
$$X_1 f_{k,m}=c_1 q^{\frac{k}{2}} f_{k,m},\qquad Y_2 f_{k,m}=c_2 q^{\frac{m}{2}} f_{k,m},$$
$$X_2 f_{k,m}=\frac{1}{c_2 q^{\frac{m}{2}}(q-q^{-1})^2}(f_{k+2,m}+f_{k-2,m}-qf_{k=2,m+4}-q^{-1}f_{k-2,m-4}),$$
$$Y_1 f_{k,m}=\frac{1}{c_1 q^{\frac{m}{2}}(q-q^{-1})^2}(f_{k,m+2}+f_{k,m-2}-qf_{k-4,m-2}-q^{-1}f_{k+4,m+2}),$$

c_1, $c_2\in\tau=(-q^{\frac{1}{2}},-1]\cup[1,q^{\frac{1}{2}})$.

Definition of the representations of $*$-algebra A_5 for $q>1$ and list of all irreducible representations see in [11].

Remark 3 If $q\in\mathbf{T}$ and the operator k_i, X_i, Y_i are bounded, then one can easily show that all irreducible representations of A_i, $i=4,5,6$ are one-dimensional. The same is true for $*$-algebra A_6, when $q\in\mathbf{R}$. It is a problem what are "integrable" representations of such $*$-algebras for unbounded operators.

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