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Conditional Symmetry and Exact Solutions of the Multidimensional Nonlinear d'Alembert Equation

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In the present paper new classes of exact solutions of the nonlinear d'Alembert equation in the space $R_{1,n},\ n \geq 2$,

$$\Box u + \lambda u^k = 0 \tag{1}$$

are constructed. Here

$$\Box = u_{00} - u_{11} - \ldots - u_{nn}, \quad u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad u = u(x),$$

$$x = (x_0, x_1, \ldots, x_n); \quad \mu, \nu = 0, 1, \ldots, n.$$

Symmetry properties of equation (1) have been studied in the paper [1] in which it was established that equation (1) is invariant under the extended Poincare algebra $A\tilde{P}(1,n)$:

$$J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_b\partial_a - x_a\partial_b, \quad P_\mu = \partial_\mu,$$

$$S = -x^\mu\partial_\mu + \frac{2u}{k-1}\partial_u \quad (a, b = 1, \ldots, n;\ \mu = 0, 1, \ldots, n).$$

The symmetry reduction of the equation (1) was considered in [1–5]. Maximal subalgebras of rank n of the algebra $A\tilde{P}(1,n)$ were described in [3, 4]. These results allow to construct all symmetry ansatzes reducing the equation (1) to ordinary differential equations. Using the solutions of the reduced equations, some classes of multiparametric exact solutions of the equation (1) were found. Classification of the maximal subalgebras of rank $n-1$ of the algebra $A\tilde{P}(1,n)$ was given in [5]. Using these subalgebras ansatzes reducing equation (1) to equations of two variables were constructed.

The present paper continues the research which was carried out in [3–5]. Here new operators of conditional symmetry of the d'Alembert equation are found. Using these operators conditionally-invariant ansatzes reducing equation (1) to ordinary differential equations are built. New classes of exact solutions of this equation are constructed.

Consider the subalgebra $L =< G_1, \ldots, G_{m-1}, P_{m+1}, \ldots, P_n >$ where $G_a = J_{0a} - J_{am}$, $a = 1, 2, \ldots, m-1$. Let solutions of the equation (1) satisfy the conditions

$$G_1 u = 0, \ldots, G_{m-1} u = 0, \quad P_{m+1} u = 0, \ldots, P_n u = 0. \tag{2}$$

Conditions (2) select from the set of solutions of the equation (1) the subset of solutions invariant under L. We investigate the symmetry of the system (1), (2).

The subalgebra L has the invariants $u, \omega_1 = x_0 - x_m, \omega_2 = x_0^2 - x_1^2 - \ldots - x_m^2$. By means of the ansatz $u = u(\omega_1, \omega_2)$ the system (1), (2) is reduced to the equation

$$\begin{gathered} 4\omega_1 u_{12} + 4\omega_2 u_{22} + 2(m+1)u_2 + \lambda u^k = 0, \\ u_{12} = \frac{\partial^2 u}{\partial \omega_1 \partial \omega_2}, \quad u_{22} = \frac{\partial^2 u}{\partial \omega_2^2}, \quad u_2 = \frac{\partial u}{\partial \omega_2}. \end{gathered} \tag{3}$$

In this case the investigation of the system (1), (2) is reduced to that of the symmetry of the equation (3).

Theorem 1. *The maximal algebra of invariance of equation (3) in the case* $k \neq 0, \frac{m+1}{m-1}$ *and* $m > 1$ *in the sense of Lie is a 4-dimensional Lie algebra* $A(4)$ *which is generated by such operators:*

$$X_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u},$$

$$X_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u}, \quad X_3 = \omega_1 \frac{\partial}{\partial \omega_2},$$

$$M = \omega_1^l \Big(\omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u} \Big), \quad l = \frac{(m-1)(k-1)}{2} - 1.$$

We classify one-dimensional subalgebras of the algebra $A(4)$ with respect to G-conjugation, where $G = \exp A(4)$.

Theorem 2. *Let* F *be a one-dimensional subalgebra of the algebra* $A(4)$*. Then* F *is conjugated with one of the following algebras:*

1) $F_1 = < X_1 + \alpha X_2 >$; 2) $F_2 = < X_2 >$: 3) $F_3 = < X_1 + \alpha X_3 >$ $(\alpha = \pm 1)$;

4) $F_4 = < X_3 >$; 5) $F_5 = < M + \alpha X_2 >$ $(\alpha = 0, \pm 1)$; 6) $F_6 = < M + \alpha X_3 >$ $(\alpha = \pm 1)$.

The following ansatzes correspond to subalgebras F_1–F_6 of Theorem 2:

$$F_1 : u = \omega_1^{\frac{\alpha+1}{1-k}} \varphi(\omega), \quad \omega = \omega_2 \omega_1^{-\alpha-1};$$

$$F_2 : u = \omega_2^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \omega_1;$$

$$F_3 : u = \omega_1^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha \ln \omega_1;$$

$$F_4 : u = \varphi(\omega), \quad \omega = \omega_1;$$

$$F_5 : u = (\omega_1^l \omega_2)^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \frac{\alpha}{l} \omega_1^{-l} + \ln \frac{\omega_2}{\omega_1};$$

$$F_6 : u = \omega_1^{\frac{1-m}{2}} \varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} + \frac{\alpha}{l} \omega_1^{-l}.$$

These ansatzes reduce the equation (3) to ordinary differential equations:

$$F_1 : -4\alpha\omega\ddot{\varphi} + \frac{4(l-\alpha k)}{k-1}\dot{\varphi} + \lambda\varphi^k = 0;$$

$$F_2 : -\frac{4\omega}{k-1}\dot{\varphi} - \frac{4l}{(k-1)^2}\varphi + \lambda\varphi^k = 0;$$

$$F_3 : -4\alpha\ddot{\varphi} + \frac{4l}{k-1}\dot{\varphi} + \lambda\varphi^k = 0;$$

$$F_4 : \lambda\varphi^k = 0;$$

$$F_5 : -4\alpha\ddot{\varphi} + \frac{4\alpha}{k-1}\dot{\varphi} + \lambda\varphi^k = 0;$$

$$F_6 : -4\alpha\ddot{\varphi} + \lambda\varphi^k = 0.$$

Using the ansatzes which correspond to the subalgebra F_1 in the case $\alpha = 0,\ \alpha = \dfrac{2l}{k+1}$ and $\alpha = \dfrac{l(k+1)}{2}$, we obtain the following solutions of equation (3):

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\{\omega_2 + C\omega_1\}, \tag{4}$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\Big\{\omega_2^{\frac{1}{2}} + C\omega_1^{\frac{2l+k+1}{2(k+1)}}\Big\}^2, \tag{5}$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\Big\{\omega_2^{\frac{1}{2}} + C\omega_1^{\frac{l(k+1)+2}{2(k+1)}}\omega_2^{\frac{k-1}{2(k+1)}}\Big\}^2. \tag{6}$$

The solution of equation (3) which is invariant under the subalgebra F_6 is

$$u^{1-k} = \frac{\lambda(k-1)^2}{8\alpha(k+1)}\omega_1^{l-1}\Big(\omega_2 + \frac{\alpha}{l}\omega_1^{1-l} + C\omega_1\Big)^2. \tag{7}$$

In order to construct new exact solutions of the d'Alembert equation, it is possible to use the operator M, acting by M on known solutions of reduced equation (3). Let θ_t be the transformation defined by the element $\exp(tM)$. The transformation θ_t acts on the variables u, ω_1 and ω_2 in the following way

$$\theta_t(\omega_i) = \omega_i(1 - lt\omega_1^l)^{-\frac{1}{l}}, \qquad \theta_t(u) = u(1 - lt\omega_1^l)^{\frac{m-1}{2l}},$$

where $i = 1, 2, ;\ l = \dfrac{(m-1)(k-1)}{2} - 1$.

Consider, for example, a one-parametric class of solution

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega_2 + C_1\omega_1)$$

of equation (3) in the case $k \neq \dfrac{m+1}{m-1}$. By means of the invariance group of equation (3) this class gives a 2-parametor class of solutions

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega_2 + C_1\omega_1)(1 + C_2\omega_1^l).$$

Thus, the classes of solutions of equations (3) are obtained. Each of these solutions expressed in the variables $x_0, x_1, \ldots, x_n$ is a solution of the d'Alembert equation (1). Acting on these solutions with the group $\widetilde{P}(1,n)$ we obtain multiparametric classes of exact solutions of the d'Alembert equation.

Using the groups of invariance of equations (1) and (3) we can do the same for solutions (5)–(7). Consequently we obtain multiparametric exact solutions of equation (1). We write down these solutions for equations (1) in the space $R_{1,n}$ using the following notations:

$$a = (a_0, a_1, \ldots, a_n), \quad b_i = (b_{i0}, b_{i1}, \ldots, b_{in}), \quad y = (y_0, y_1, \ldots, y_n),$$

$$y_\mu = x_\mu + \alpha_\mu, \quad a \bullet y = a_0 y_0 - a_1 y_1 - \ldots - a_n y_n, \quad \sigma_m = \frac{\lambda(k-1)^2}{2(m-1)(k-1)-4}.$$

1) $$u^{1-k} = \sigma_m (y \bullet y + (b_1 \bullet y)^2 + \ldots + (b_{n-m} \bullet y)^2)\Big(1 + c(a \bullet y)^{\frac{(k-1)(m-1)}{2}-1}\Big),$$

$$a \bullet a = 0, \ a \bullet b_i = 0, \ b_i \bullet b_i = -1, \ b_i \bullet b_j = 0$$

$$\text{if} \ \ i \neq j \ (i, j = 1, 2, \ldots, n-m); \ k \neq \frac{m+1}{m-1}, \ m = 2, 3, \ldots, n.$$

2) $$u^{1-k} = \sigma_m \Big\{ \Big[(y \bullet y + (b_1 \bullet y)^2 + \ldots + (b_{n-m} \bullet y)^2)\Big(1 + C_1 (a \bullet y)^{\frac{(k-1)(m-1)}{2}-1}\Big)\Big]^{\frac{1}{2}} +$$
$$C_2 (a \bullet y)^{\frac{m(k-1)}{2(k+1)}} \Big(1 + C_1 (a \bullet y)^{\frac{(k-1)(m-1)}{2}-1}\Big)^{\frac{k-1}{2(k+1)}} \Big\}^2,$$

$$a \bullet a = 0, \quad a \bullet b_i = 0, \quad b_i \bullet b_i = -1, \quad b_i \bullet b_j = 0$$

$$\text{if} \ \ i \neq j \ (i, j = 1, 2, \ldots, n-m); \quad k \neq \frac{m+1}{m-1}, \ m = 2, 3, \ldots, n.$$

3) $$u^{1-k} = \sigma_m \Big\{ \Big[(y \bullet y + (b_1 \bullet y)^2 + \ldots + (b_{n-m} \bullet y)^2)\Big(1 + C_1 (a \bullet y)^{\frac{(k-1)(m-1)}{2}-1}\Big)\Big]^{\frac{1}{2}} +$$
$$C_2 (a \bullet y)^{\frac{(k-1)}{4(k+1)}((k+1)(m-1)-2)} (y \bullet y + (b_1 \bullet y)^2 + \ldots + (b_{n-m} \bullet y)^2)^{\frac{k-1}{2(k+1)}} \Big\}^2,$$

$$a \bullet a = 0, \quad a \bullet b_i = 0, \quad b_i \bullet b_i = -1, \quad b_i \bullet b_j = 0$$

$$\text{if} \ \ i \neq j \ \ (i, j = 1, 2, \ldots, n-m); \ \ k \neq \frac{m+1}{m-1}, \ m = 2, 3, \ldots, n.$$

4) $$u^{1-k} = \frac{\lambda(k-1)^2}{8\alpha(k+1)} (a \bullet y)^{\frac{(k-1)(m-1)}{2}-2} \Big\{ y \bullet y + (b_1 \bullet y)^2 + \ldots +$$
$$(b_{n-m} \bullet y)^2 + \frac{2\alpha}{(k-1)(m-1)-2} (a \bullet y)^{2-\frac{(k-1)(m-1)}{2}} \Big\}^2,$$

$$a \bullet a = 0, \quad a \bullet b_i = 0, \quad b_i \bullet b_i = -1, \quad b_i \bullet b_j = 0$$

$$\text{if} \ \ i \neq j \ \ (i, j = 1, 2, \ldots, n-m); \ \ k \neq -1, \frac{m+1}{m-1}, \ \ m = 2, 3, \ldots, n.$$

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