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Nonlinear Maxwell Equations

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Abstract

The infinite series of Lorentz and Poincaré-invariant nonlinear versions of the Maxwell equations are suggested. Some properties of these equations are considered.

Nonlinear equations of theoretical and mathematical physics attract considerable attention because of their specific properties, such as absence of the superposition principle, nonlinear fields interactions, existence of soliton solutions. Nonlinear equations in electrodynamics were suggested for the first time by Born [1], Born and Infeld [2], and also Schrödinger [3] and were derived from the variational principle. Later Fushchych and Tsifra [4], Fushchych [5], Fushchych, Tsyfra and Boyko [6] have applied theoretical-algebraic approach to this problem. The purpose of the present paper is formulation of Lorentz and Poincaré-invariant equations with the help of the variable replacement method. Let us introduce one-dimensional Lorentz transformations [7]

$$x_1' = \frac{x_1 - \beta ct}{\sqrt{1-\beta^2}}; \quad x_2' = x_2; \quad x_3' = x_3; \quad t' = \frac{t - \beta x_1/c}{\sqrt{1-\beta^2}}. \tag{1}$$

Here $x_{1,2,3} = x, y, z$; c is the speed of light; t is the time; $\beta = V/c$; V is the velocity of movement of inertial frame K' relative to K.

One can see by calculation that the nonlinear equations

$$\Phi_1(I_1, I_2)\nabla.\,\mathbf{E} = 4\pi\rho; \quad \Phi_1(I_1, I_2)(\nabla \times \mathbf{H} - \frac{1}{c}\partial_t \mathbf{E}) = +4\pi\rho\mathbf{v};$$

$$\Phi_2(I_1, I_2)\nabla.\,\mathbf{H} = 4\pi\mu; \quad \Phi_2(I_1, I_2)(\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{H}) = -4\pi\mu\mathbf{w} \tag{2}$$

are invariant with respect to transformations (1), if the variables entering into them are transformed in the following way [7]:

$$E_1' = E_1; \quad E_2' = \frac{E_2 - \beta H_3}{\sqrt{1-\beta^2}}; \quad E_3' = \frac{E_3 + \beta H_2}{\sqrt{1-\beta^2}}; \tag{3}$$

$$H_1' = H_1; \quad H_2' = \frac{H_2 + \beta E_3}{\sqrt{1-\beta^2}}; \quad H_3' = \frac{H_3 - \beta E_2}{\sqrt{1-\beta^2}}; \tag{4}$$

$$\rho' = \rho \frac{1 - v_1 V/c^2}{\sqrt{1-\beta^2}}; \tag{5}$$

$$\mu' = \mu \frac{1 - w_1 V/c^2}{\sqrt{1-\beta^2}}; \tag{6}$$

$$v'_1 = \frac{v_1 - V}{1 - v_1 V/c^2}; \quad v'_{2,3} = v_{2,3} \frac{\sqrt{1-\beta^2}}{1 - v_1 V/c^2}; \tag{7}$$

$$w'_1 = \frac{w_1 - V}{1 - w_1 V/c^2}; \quad w'_{2,3} = w_{2,3} \frac{\sqrt{1-\beta^2}}{1 - w_1 V/c^2}. \tag{8}$$

Here E, H are electric and magnetic fields; ρ, μ are densities of electromagnetic charges; v, w are charge velocities; Φ_1, Φ_2 are arbitrary functions of Lorentz and Poincaré invariants of fields $I_1 = 2(\mathbf{E}^2 - \mathbf{H}^2)$, $I_2 = (\mathbf{E}.\mathbf{H})^2$ [3]. In proving the invariance of the equations it is necessary to take into account the following: invariance of the speed of light; the transformation properties of electromagnetic fields and charge densities; the law of transformation of velocities; invariance of the functions Φ_1 and Φ_2.

Because of arbitrariness of the functions Φ_1 and Φ_2, system (2) contains an infinite set of particular realizations of nonlinear Maxwell equations, among which it is possible to indicate the following basic versions:

- the linear free Maxwell equations [3] with $\rho = \mu = 0$;
- the linear one-charge Maxwell equations [3] with $\Phi_1 = \Phi_2 = 1, \ \mu = 0$;
- the linear two-charge Maxwell equations [8] with $\Phi_1 = \Phi_2 = 1$.

Let us note some general properties of these nonlinear equations induced.

The equations (2) become not only relativistic but also conformally invariant if the functions are $\Phi_1(I_1^2/I_2)$, $\Phi_2(I_1^2/I_2)$. The statement results from the proof of conformal symmetry of linear Maxwell equations and identical conformal dimensions of the values I_1^2 and I_2.

Equations (2) become linear in absence of currents and charges and so contain the classical electrodynamics of free fields.

Generally, the nonlinearity is conditioned by currents and charges.

The equations keep the possibility of a electromagnetic field definition through the two-potentials $A^a = (\phi, \mathbf{A})$, $B^a = (\Phi, \mathbf{B})$, $a = 0, 1, 2, 3$ [9]

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}/c - \nabla \times \mathbf{B}; \quad \mathbf{H} = -\nabla\Phi - \partial_t \mathbf{B}/c + \nabla \times \mathbf{A}. \tag{9}$$

The two-potentials satisfy the nonlinear D'Alembert equations

$$\Phi_1(I_1, I_2)\Box A^a = 4\pi J^a; \ \Phi_2(I_1, I_2)\Box B^a = 4\pi K^a \tag{10}$$

under condition of the relativistic invariant calibrations $\partial_a A^a = 0$, $\partial_a B^a = 0$, where $\partial_a = \partial/\partial x^a$, $x^0 = ct$, $x^{1,2,3} = x, y, z$; $g_{ab} = diag(+, -, -, -)$; $J^0 = \rho$, $\mathbf{J} = \rho\mathbf{v}/c$; $K^0 = \mu$, $\mathbf{K} = \mu\mathbf{w}/c$. Similarly to the initial equations (2), the free equations (10) automatically become linear.

In the important particular case of electrostatic charges in the one-charge electrodynamics with $\mathbf{A} = 0$, the scalar potential ϕ satisfies the nonlinear Laplace-Poisson equation

$$\Phi_1((\nabla\phi)^2) \triangle \phi = -4\pi\rho(x). \tag{11}$$

Putting here Fourier-decomposition of the potential $\phi = (2\pi)^{-3} \int \phi_k \exp(i\mathbf{k}.\mathbf{x}) d^3k$ in the case of electrical charge with the density $\rho(x)$ for the component ϕ_k, we have

$$\phi_k = (4\pi/k^2) \int \rho(x) F_1((\nabla\phi)^2) \exp(-\mathbf{k}.\mathbf{x}) d^3x = (4\pi/k^2)(\rho F_1)_k. \tag{12}$$

Here $(\rho F_1)_k$ means the form-factor characterizing the electricity distribution in the effective charge $Q = \int \rho F_1 d^3x$, $F_1 = 1/\Phi_1$. The form-factor can differ from unit. This will mean the availability of corrections to the Coulomb field of charge.

Let us write the equations (2) in a different form. We divide their right parts into the functions Φ_1 and Φ_2, designate $1/\Phi_1 = F_1$, $1/\Phi_2 = F_2$ and instead of $\rho(x)$, $\mu(x)$, $\mathbf{J} = \rho(x)\mathbf{v}/c$, $\mathbf{K} = \mu(x)\mathbf{w}/c$ we take new variables

$$\rho \to \rho(x)F_1; \quad \mu \to \mu(x)F_2; \quad \mathbf{J} \to \rho(x)F_1\mathbf{v}/c; \quad \mathbf{K} \to \mu(x)F_2\mathbf{w}/c. \tag{13}$$

We will refer to the densities of charges and currents ρ, μ, $\mathbf{J}$ and $\mathbf{K}$ as the initial ones, and to the values corresponding to them as the effective ones. Then it is possible to say that nonlinear microscopic equations of electrodynamics are equations which contain the effective values of charge densities and current densities instead of the initial ones [10]

$$\nabla.\mathbf{E} = 4\pi F_1(I_1, I_2)\rho; \quad \nabla \times \mathbf{H} - \frac{1}{c}\partial_t \mathbf{E} = 4\pi F_1(I_1, I_2)\rho\frac{\mathbf{v}}{c};$$

$$\nabla.\mathbf{H} = 4\pi F_2(I_1, I_2)\mu; \quad \nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{H} = -4\pi F_2(I_1, I_2)\mu\frac{\mathbf{w}}{c}; \tag{14}$$

$$\Box A^a = 4\pi F_1 \mathbf{J}^a; \quad \Box B^a = 4\pi F_2 \mathbf{K}^a. \tag{15}$$

These equations realize the principle of self-action: the initial charges generate electromagnetic fields which in its turn influence the initial charges, their densities and sizes up to reaching the equilibrium state with the generating fields. So, in nonlinear versions of the Maxwell and D'Alembert equations (14), (15) the electromagnetic charges

$$Q = \int \rho(x) F_1(I_1, I_2) d^3x; \quad P = \int \mu(x) F_2(I_1, I_2) d^3x \tag{16}$$

receive at least partly the field nature. This property of charge is absent in the linear electrodynamics. The effective charges Q and P keep the property of Lorentz-invariance owing to the invariance of functions F_1 and F_2, and are integrals of movement due to the existence of the continuity equations

$$\partial_t(\rho F_1) + c\nabla.(F_1\mathbf{J}) = 0; \quad \partial_t(\mu F_2) + c\nabla.(F_2\mathbf{K}) = 0. \tag{17}$$

It follows from the equations (17) that the initial charges are not conserved. For example, for the electric charge $q = \int \rho d^3x$ we have

$$\partial_t q = -\oint \rho \mathbf{v} ds - \int (\partial_t F_1 + \mathbf{v}.\nabla F_1)(\rho/F_1) d^3x. \tag{18}$$

Here ds is an element of the area surrounding the volume element d^3x as usual. The change of the charge q is conditioned not only by the density of current $\mathbf{j} = \rho\mathbf{v}$, but by the change of the field invariant $F_1(I_1, I_2)$ in time and space.

As far as the nonlinear Maxwell equations satisfy the requirement of relativistic invariance, they have the potential interest to physics. In addition to the known general theoretical questions of electrical charge stability and nature of its mass [3], it is possible to point out also the field nature of a charge and the necessity of experimental verification of the Coulomb law at short distances. The existence of equations (14) prompts us also to induce the relativistic invariant action integral in the case of the one-charge electrodynamics in a more general form

$$S = -mc\int \Psi_1(I_1, I_2)ds - \frac{1}{c}\int \Psi_2(I_1, I_2)A_a J^a d^4x - \frac{1}{16\pi c}\int \Psi_3(I_1, I_2)I_1 d^4x. \quad (19)$$

Here as usually, m is the rest mass of a particle, ds is the element of the interval, $A_a = (\phi, -\mathbf{A})$, $d^4x = cdtdxdydz$ [7], Ψ_1, Ψ_2, Ψ_3 are the functions of relativistic invariants I_1 and I_2.

According to (19) we can indicate six versions of Maxwell electrodynamics with the invariant speed of light:

- the classical linear electrodynamics with $\Psi_1 = \Psi_2 = \Psi_3 = 1$ [3], [7];
- the linear electrodynamics with $\Psi_1 \neq 1$, $\Psi_2 = \Psi_3 = 1$;
- the nonlinear electrodynamics of the first type with $\Psi_3 \neq 1$, $\Psi_1 = \Psi_2 = 1$;
- the nonlinear electrodynamics of the second type with $\Psi_2 \neq 1$, $\Psi_1 = \Psi_3 = 1$;
- the nonlinear electrodynamics of the third type with $\Psi_2 \neq 1, \Psi_3 \neq 1, \Psi_1 = 1$;
- the nonlinear electrodynamics of the fourth type with all functions $\Psi \neq 1$.

In particular, the Born model with $\Psi_3 = 4E_0^2[1 - (1 - I_1/2E_0^2)^{1/2}]/I_1$ [1], the Born-Infeld model with $\Psi_3 = 4E_0^2[1 - (1 - I_1/2E_0^2 - I_2/4E_0^4)^{1/2}]/I_1$ [2], the Schrödinger model with $\Psi_3 = 2E_0^2\, ln(1 + I_1/2E_0^2)$ [3] belong to the nonlinear version of the first type. (Here E_0 is the maximum field [3]).

This work belongs to the nonlinear version of the second type, as far as the variation of the integral (19) with the constant value of $\Psi_2 J^a$ leads to the equations (14). For example, within the framework of this version the nonlinear Laplace-Poisson equation (11) may be written as follows:

$$\left[1 + \alpha\Big(\frac{\partial\phi}{\partial r}\Big)^2\right]\ \left[\Big(\frac{1}{r^2}\Big)\frac{\partial}{\partial r}\Big(r^2\frac{\partial\phi}{\partial r}\Big)\right] = \begin{cases} -\dfrac{2q}{r^2}\Big(\dfrac{a}{\sqrt{\pi}}e^{-a^2r^2}\Big) & \text{if} \quad \rho = \rho_1; \\ -\dfrac{2q}{r^2}\Big(\dfrac{a}{\pi}\dfrac{sinar}{ar}\Big) & \text{if} \quad \rho = \rho_2 \end{cases}. \quad (20)$$

Here we put $\Phi_1 = [1 + \alpha I_1] = [1 + \alpha\mathbf{E}^2]_{H=0} = [1 + \alpha(\partial\phi/\partial r)^2]$; $\rho_1 = (q/2\pi r^2)(a/\sqrt{\pi}) \times \exp(-a^2r^2)$; $\rho_2 = (q/2\pi r^2)(a/\pi)(sinar/ar)$, q is the electrical charge, $r = (x^2+y^2+z^2)^{1/2}$, $\alpha = k/a$, k is the proportionality coefficient, a is the parameter with inverse length dimension (1/cm). Tending $a \to \infty$ ($\alpha \to 0$), we have the linear Laplace-Poisson equation

$$\frac{1}{r^2}\ \frac{\partial}{\partial r}\ \Big(r^2\ \frac{\partial\phi}{\partial r}\Big) = -\frac{2q}{r^2}\,\delta\,(r), \quad (21)$$

where a solution of the equation has the known form $\phi = q/r$. One can see that the linear theory is one with point charges. The nonlinear theory is one with the density of charges distributed over space. Densities of charges ρ_1 and ρ_2 correspond to the various physical models of charges.

The other versions were not investigated. They can be accompanied by dependence of the effective mass on electromagnetic field.

In addition to these six versions, it is possible to formulate the nonlinear electrodynamics with a variable velocity of light. The model of this type was proposed by Fushchych [5].

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