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Derivation of asymptotical formulas for resolution of systems of differential equations with symmetrical matrices

M.I. SHKIL', P.F. SAMUSENKO

Drahomanov Pedagogical University,
9 Pyrogov Street, Kyïv, Ukraïna

Abstract

Asymptotic formulae for resolution of L-diagonal systems of ordinary differential equations with symmetrical matrices are derived.

1) We shall consider a system of linear differential equations

$$\frac{dx}{dt} = (\Lambda(t) + Q(t))x, \tag{1}$$

where x is an n-vector, $\Lambda(t)$ is an $(n \times n)$ diagonal matrix, and $Q(t)$ is an $(n \times n)$-matrix with summable elements in the interval (t_0, ∞). Such a system was called L-diagonal system by I.M. Rapoport [1]. Studying these systems we assume that:

a) The elements $\omega_i(t)$ $(i = 1, 2, ..., n)$ of the diagonal matrix $\Lambda(t)$ are summable in the interval (t_0, t_1) for any finite t_1;

b) There exists T_0 big enough for any difference

$$\operatorname{Re}\omega_i(t) - \operatorname{Re}\omega_j(t), \quad i, j = 1, ..., n,$$

not to change sign for $t \geq T_0$.

Then system (1) can be solved for $t \geq t_0$ and its n particular solutions have the form

$$x_i = \eta_{ij}(t) \exp \int_{t_0}^{t} \omega_j(t) dt, \quad i, j = 1, 2, ..., n,$$

where $\eta_{ij}(t)$ are continuous functions in the closed interval $[t_0, \infty]$, and $\eta_{ij}(\infty) = 0$ when $i \neq j, \eta_{jj}(\infty) = 1$. I.M. Rapoport [1] found substitutions which can help us to reduce systems of differential equations

$$\frac{dx}{dt} = A(t)x \tag{2}$$

to L-diagonal systems in the case when roots of the characteristic equation are simple. In the paper [2] we suggested the method for construction of the mentioned substitutions in the case when the roots of the characteristic equation for $t \geq t_0$ maintain constant multiplicity.

2) In this paper we suggest the method for construction of such substitutions for systems of differential equations

$$B(t)\frac{dx}{dt} = A(t)x, \tag{3}$$

where the matrix $B(t)$ may not have an inverse matrix, when $t \geq t_0$.

3) We will consider instead of the system (3) the system

$$\varepsilon B\,(t)\ \frac{dx}{dt} = A(t)x, \tag{3'}$$

where $\varepsilon > 0$ is a real parameter.

The system $(3')$ coincides with the system (3) when $\varepsilon = 1$. We shall make use of a substitution $t = \varepsilon t_1$ in the system $(3')$. Now we have the system

$$B(t)\frac{dx}{dt_1} = A(t)x. \tag{4}$$

For construction of a fundamental matrix of solutions for this system, we can use the method from [2].

The equation

$$\det(A(t) - \lambda B(t)) = 0 \tag{5}$$

has υ roots $\lambda(t), ..., \lambda_\upsilon(t)\ \ (\upsilon \leq n)$.

We assume that the matrices $A(t), B(t)$ are symmetric for $t \geq t_0$. Then the roots of the equation (5) are real [3]. We assume that the roots $\lambda_i(t)$ are different when $t \leq t < \infty$. Hence when $t \geq t_0 \qquad \lambda_i(t) \neq \lambda_j(t),\ \ i \neq j,\ \ i, j = 1, ..., \upsilon$, so we can construct proper vectors $\mu_i(t)$ of the matrix $A(t)$ with respect to the matrix $B(t)$ in order for the scalar product

$$(B(t)\mu_i(t), \mu_j(t)) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \quad i, j = 1, ..., \upsilon. \end{cases}$$

We put

$$x = \mathcal{U}_m(t, \varepsilon)y, \qquad \mathcal{U}_m(t, \varepsilon) = \sum_{s=0}^{m} \varepsilon^s \mathcal{U}_s(t),$$

where y is an n-measurable vector, and $\mathcal{U}_s(t)$ are square $(n \times n)$-matrices. We have

$$B(t)\,\mathcal{U}_m(t, \varepsilon)\frac{dy}{dt_1} = (A(t)\,\mathcal{U}_m(t, \varepsilon) - \varepsilon B(t)\,\mathcal{U}_m^{'}(t, \varepsilon))y.$$

We construct the matrices $\mathcal{U}_s(t)\ \ (s = 0, 1, ..., m)$ so as to get the matrix equality

$$A(t)\,\mathcal{U}_m(t, \varepsilon) - \varepsilon B(t)\,\mathcal{U}_m^{'}(t, \varepsilon) = B(t)\,\mathcal{U}_m(t, \varepsilon)(\Lambda(t, \varepsilon) + \varepsilon^{m+1} C_m(t, \varepsilon)), \tag{6}$$

where $\Lambda(t, \varepsilon)$ is a diagonal matrix,

$$\Lambda_m(t, \varepsilon) = \sum_{s=0}^{m} \varepsilon^s \Lambda_s(t).$$

We have to compare coefficients of $\varepsilon^0, \varepsilon^1, ..., \varepsilon^m$ in the matrix equalities (6). We have a matrix system of equations

$$A(t)\mathcal{U}_0(t) - B(t)\mathcal{U}_0(t)\Lambda_0(t) = 0, \tag{7}$$

$$A(t)\mathcal{U}_s(t) - B(t)\mathcal{U}_s(t)\Lambda_0(t) = B(t)\mathcal{U}_{s-1}^{'}(t) + B(t)\sum_{j=1}^{s}\mathcal{U}_{s-j}(t)\Lambda_j(t). \tag{8}$$

Let $\Lambda_0(t) = \operatorname{diag}\{\lambda - 1(t), \lambda_2(t), ..., \lambda_\upsilon(t), ..., \lambda_n(t)\}$.

We write the matrix equation (7) in the vector form and designate columns of the matrix $\mathcal{U}_0(t)$ by $u_{0i}(t)$ $(i = 1, 2, ..., n)$. We have

$$(A(t) - \lambda_j(t)B(t))u_{0i}(t) = 0.$$

Then

$$u_{oi}(t) = \mu_i(t) \ \ (i = 1, 2, ..., n) \ \text{ and } \ u_{0i}(t) \equiv 0 \ \ (i = \upsilon + 1, ..., n).$$

Let us consider (8) when $s = 1$:

$$A(t)\mathcal{U}_1(t) - B(t)\mathcal{U}_1(t)\Lambda_0(t) = B(t)\mathcal{U}_0^{'}(t) + B(t)\mathcal{U}_0(t)\Lambda_1(t);$$

or in the vector form

$$(A(t) - \lambda_i(t)B(t))u_{1i}(t) = B(t)u_{0i}^{'}(t) + B(t)u_{0i}(t)\lambda_{1i}(t) \equiv g_{1i}(t), \quad i = 1, 2, ..., n. \tag{9}$$

The equation (9) can be solved relatively $u_{1i}(t)$ iff, when the vector $g_{1i}(t)$ $(i = 1, 2, ..., n)$ is orthogonal to the vector which is a solution of the conjugated system corresponding to the homogeneous system (9). $A(t)$ and $B(t)$ are symmetric, so the conjugated system coincides with (9). So (9) has a solution iff, when

$$(g_{1i}(t), \mu_i(t)) = 0 \tag{10}$$

for all $t \geq t_0$. For $i = 1, ..., n$ we have

$$(B(t)u_{0i}^{'}(t), \mu_i(t)) + (B(t)u_{0i}(t)\lambda_{1i}(t), \mu_i(t)) = 0$$

or

$$\lambda_{1i}(t) = -(B(t)u_{oi}^{'}(t), \mu_i(t)), \quad i = 1, 2, ..., n.$$

If $i = \upsilon + 1, ..., n$, then (10) will change to an identity. So we can take $\lambda_{1i}(t) \equiv 0$ $(i = \upsilon + 1, ..., n)$. Thus substituting the obtained values of $\lambda_{1i}(t)$ $(i = 1, ..., n)$ to the system (9), we get the vector-column of the first part is orthogonal for all $t \geq t_0$ to a non–trivial solution of the conjugate system. We look for this solution in such a form

$$u_{1i}(t) = \sum_{r=1}^{\upsilon} c_{ri}^{(1)}(t)\mu_r(t), \quad i = 1, ..., \upsilon, \tag{11}$$

where $c_{ri}^{(1)}(t)$ is a function which must be defined for the vector (11) to satisfy the system (9).

Substituting (11) to system (9) and multiplying this result by the vector $\mu_j(t)$ $(j = 1,...,\upsilon)$, we have

$$c_{ji}^{(1)}(t)(\lambda_j(t)-\lambda_i(t)) = (g_{1i}(t),\mu_j(t)), \quad j=1,...,\upsilon.$$

When $i=j$ we have the identity $c_{jj}^{(1)}(t)\cdot 0 \equiv 0$. Hence we can take any function $c_{jj}^{(1)}(t)$. We take $c_{jj}^{(1)}(t)\equiv 0$, for $t\geq t_0$. When $i\neq j$

$$c_{ji}^{(1)}(t) = \frac{(g_{1i}(t),\mu_j(t))}{\lambda_j(t)-\lambda_i(t)}.$$

Then

$$u_{1i}(t) = \sum_{r=1}^{\upsilon}\frac{(g_{1i}(t),\mu_j(t))}{\lambda_j(t)-\lambda_i(t)}\mu_r(t).$$

We assume that $u_{1i}(t)\equiv 0, \quad i=\upsilon+1,...,n.$

Thus we defined the vectors $u_{1i}(t)$ $(i=1,2,...,n)$ (in the matrix $\mathcal{U}_1(t)$) and functions $\lambda_{1i}(t)$ $(i=1,2,...,n)$ (in the matrix $\Lambda_1(t)$). Using the method of mathematical induction, we can find from equations (8) all the following matrices $\mathcal{U}_s(t)$ and $\Lambda_s(t)$ $(s=2,3,...,m)$. So the system (4) has the form

$$B(t)\mathcal{U}_m(t,\varepsilon)\frac{dy}{dt_1} = B(t)\mathcal{U}_m(t,\varepsilon)(\Lambda_m(t,\varepsilon)+\varepsilon^{m+1}C_m(t,\varepsilon))y. \tag{12}$$

We can find the matrix $C_m(t,\varepsilon)$ from (6)

$$\varepsilon^{m+1}B(t)\mathcal{U}_m(t,\varepsilon)C_m(t,\varepsilon) = A(t)\mathcal{U}_m(t,\varepsilon)-\varepsilon B(t)\mathcal{U}_m'(t,\varepsilon)-B(t)\mathcal{U}_m(t,\varepsilon)\Lambda_m(t,\varepsilon).$$

We assume that

$$(E-B(t)\mathcal{U}_m(t,1)(B(t)\mathcal{U}_m(t,1))^-)D_m(t,1)=0,$$

is a true equality for $\varepsilon=1$ and $t_0\leq t<+\infty$, where $(B(t)\mathcal{U}_m(t,1))^-$ is a half-inverse matrix for the matrix $B(t)\mathcal{U}_m(t,1)$,

$$D_m(t,1) = -B(t)\left(\mathcal{U}_m'(t,1)+\sum_{r=1}^{m}\sum_{j=r}^{m}\mathcal{U}_j(t)\Lambda_{m+r-j}(t)\right).$$

Then

$$C_m(t,1) = (B(t)\mathcal{U}_m(t,1))^-D_m(t,1).$$

So, we have

$$B(t)\mathcal{U}_m(t,1)\frac{dy}{dt} = B(t)\mathcal{U}_m(t,1)(\Lambda_m(t,1)+C_m(t,1))y.$$

Let the system

$$\frac{dy}{dt} = (\Lambda_m(t,1)+C_m(t,1))y, \quad t\geq t_0 \tag{13}$$

be L-diagonalizable. We can understand that every solution of (13) is a solution of (12). With the condition $x = \mathcal{U}_m(t,1)y$ we can find solutions of the system (3). Looking at the conditions for matrices $A(t), B(t)$, when $t \geq t_0$, we receive solutions of (3) in such a form

$$x_j = \mu_{ij}(t) \exp \int_{t_0}^{t} \omega_j(t)dt, \quad i,j = 1,2,...,n,$$

where $\mu_{ij}(t)$ are continuous functions on the interval $[t_0, \infty)$.

Theorem 1. *For the system (3) the following is true:*

1) matrices $A(t)$ and $B(t)$ on the interval $[t_0, \infty)$ have continuous derivatives;

2) $A(t), B(t)$ are symmetric when $t \geq t_0$;

3) roots $\lambda_i(t)$ $(i = 1,2,...,\upsilon)$ of the equation

$$\det(A(t) - \lambda B(t)) = 0,$$

when $t \geq t_0$, are simple;

4) when $\varepsilon = 1$ and $t_0 \leq t < +\infty$ the equality

$$(E - B(t)\mathcal{U}_m(t,1)(B(t)\mathcal{U}_m(t,1))^{-})D_m(t,1) = 0,$$

holds, where $\mathcal{U}_m(t,1), D_m(t,1)$ are the matrices that we have found.

If the system (13) is a L-diagonal system, then n particular solutions of system (3) have the form

$$x_i = \mu_{ij}(t) \exp \int_{t_0}^{t} \omega_j(t)dt, \qquad i,j = 1,2,...,n,$$

where $\mu_{ij}(t)$ are continuous functions on the interval $[t_0, \infty)$.

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