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On Symmetry of the Generalized Breit Equation

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Abstract

In this paper we find the complete set of symmetry operators for the two-particle Breit equation in the class of first-order differential operators with matrix coefficients. A new integral of motion is obtained.

Two-particle equations play an important role in quantum electrodynamics [1] and quark and proton models [2], therefore, the problem to survey their symmetry is of a great interest.

In this work, the complete set of symmetry operators of the two-particle Breit equation in the class of differential operators of the first order with matrix coefficients ($\mathcal{M}_1$ class) is found. The new integral of motion of this equation is also found, which is not linked with the classical (Lie's) symmetry.

1. The Breit equation [3], which describes the system of two interacting particles with spins $s_{(1)} = s_{(2)} = \frac{1}{2}$ and mass $m_{(1)}, m_{(2)}$, has the following form [1]

$$L\Psi = 0, \quad L = i\frac{\partial}{\partial t} - H_{(1)} - H_{(2)} - V, \tag{1}$$

where $\Psi \equiv \Psi(x_0, x_{(1)}, x_{(2)})$ is a 16 component wave function,

$$H_{(k)} = \bar{\alpha}_{(k)} \cdot \bar{\rho}_{(k)} - \beta_{(k)} m_{(k)}; \qquad \bar{\rho}^{(k)} = -i\frac{\partial}{\partial x^{(k)}}, \quad k = 1, 2; \tag{2}$$

$$\bar{\alpha}_{(k)} = (\alpha^1_{(k)}, \alpha^2_{(k)}, \alpha^3_{(k)}); \quad \alpha^k_{(k)} = \gamma_0^{(k)}\gamma_a^{(k)}; \quad \beta_{(k)} = \gamma_0^{(k)};$$

$\{\gamma_0^{(1)}\gamma_a^{(1)}\}$ and $\{\gamma_0^{(2)}\gamma_a^{(2)}\}$ are two commutative sets of Dirac matrices, V is the interaction potential (Breit potential)

$$V = V_1 - V_2\bar{\alpha}^{(1)}\bar{\alpha}^{(2)} + V_3\frac{1}{x^2}(\bar{\alpha}^{(1)} \cdot \bar{x})(\bar{\alpha}^{(2)} \cdot \bar{x}),$$

$$V_1 = V_2 = V_3 = \frac{e_{(1)}e_{(2)}}{x}, \quad \bar{x} = \bar{x}_{(1)} - \bar{x}_{(2)}, x = \mid x \mid, \tag{3}$$

$e_{(i)}$ is a charge of the particle with number i.

The equation (1) plays an important role in quantum electrodynamics [1] and (with another potential) is successfully used for description of a meson mass spectrum [4, 5]. Maximum symmetry (in the Lie's sense) of the Breit equation is described by the following theorem.

Theorem 1. *Maximum group of invariance of the Breit equation is the semiparametric Lie group, the generators of which have the following form*

$$\rho_0 = i\frac{\partial}{\partial x_0}, \quad \rho_a = \rho_a^{(1)} + \rho_a^{(2)}, \quad J_a = \varepsilon_{abc}\Big(x_b^{(1)}\rho^{(1)} + x_b^{(2)}\rho_c^{(2)}\Big) + S_a, \tag{4}$$

where

$$S_a = \frac{i}{4}\varepsilon_{abc}\Big(\gamma_b^{(1)}\gamma_c^{(1)} + \gamma_b^{(2)}\gamma_c^{(2)}\Big). \tag{5}$$

The proof is cumbersome, therefore we only present its scheme. The description of symmetry generators Q of the group of equation (1) is reduced to solution of the equation [6]

$$[Q, L]\Psi = \alpha_Q L\Psi, \tag{6}$$

where L is operator (1), $\{\Psi\}$ is the set of solutions of equation (1), $Q = A_0\rho_0 + A_a^{(1)}\rho_a^{(1)} + A_a^{(2)}\rho_a^{(2)} + B$; $A_0, A_a^{(1)}, A_a^{(2)}$ are unknown functions of $x_0, x^{(1)}, x^{(2)}$; α_Q, B are 16×16 matrices, in general case depending on $x_0, x^{(1)}$ and $x^{(2)}$. Expanding unknown matrices α_Q and B on the complete of the set of matrices system $\Gamma_A^{(1)}\Gamma_B^{(2)}$, $A, B = 1, 2, \ldots, 16$, where $\{\Gamma_A^{(i)}\} = \{I, \gamma_k^{(i)}, \gamma_k^{(i)}, \gamma_L^{(i)}\}, k, L = 0, 1, 2, 3, 4$, by easy but cumbersome calculations we receive the general solution for Q in the form of a linear combination of operators (4) and the trivial single operator.

Thus, (Lie) classical symmetry of the Breit equation is rather restricted, this equation is invariant neither under Lorentz transformations nor Galilei ones and so does not satisfy any principle of relativity. Two-particle equations are invariant under the Galilei group and describe a pair of the interacting particles with the same accuracy as equation (1), [7].

2. For many applications (for example, for description of the coordinate system where there are solutions in the separating variables), the knowledge of symmetry operators from wider classes than generators of symmetry groups is necessary [8, 9]. Here we find the complete set of symmetry operators of the Breit equation in the class $\mathcal{M}_1$ of differential operators of the first order with matrix coefficients, which satisfy conditions (6).

We limit ourselves to the equation in the center-mass-system $p_{(1)} + p_{(2)} = 0, p_{(1)} = -p_{(2)} = p$.

Theorem 2. *A complete set of symmetry operators in the class $\mathcal{M}_1$ for the Breit equation consists of four operators $J_a, H (a = 1, 2, 3)$ for $m_{(1)} \neq m_{(2)}$, and 15 operators $\{Q_b, J_aQ_b, HQ_b\}$ $(b = 1, 2, 3)$ for $m_{(1)} = m_{(2)}$, where*

$$H = (\bar\alpha_{(2)} - \bar\alpha_{(1)})\bar p + \beta_{(1)}m_{(1)} + \beta_{(2)}m_{(2)} + V; \quad J_a = \varepsilon_{abc}x_b p_c + S_a,$$

$$Q_1 = \frac{1}{16}(I - \gamma_k^{(1)}\gamma_k^{(2)})^2; \quad Q_2 = (I - \gamma_k^{(1)}\gamma_k^{(2)})\gamma_4^{(1)}\gamma_4^{(2)}; \quad Q_3 = I - Q_1 - Q_2. \tag{7}$$

V and S_a are given in (3),(5), $k = 0, 1, 2, 3; \gamma_4^{(l)} = \gamma_0^{(l)}\gamma_1^{(l)}\gamma_2^{(l)}\gamma_3^{(l)}$, $l = 1, 2$.

The proof can be carried out according to the same scheme as Theorem 1.

In case $m_{(1)} \neq m_{(2)}$, a complete set of symmetry operators $Q \in \mathcal{M}_1$ reduces to generators (4) where $p_a \equiv 0$. For $m_{(1)} = m_{(2)}$ the symmetry operators additionally include matrices Q_a and their products with p_0 and J_a. Symmetry operators Q_a are orthoprojectors and stipulate the possibility of Breit equation splitting at $m_{(1)} = m_{(2)}$ for three unsplitting subsystems for the 10, 5 and 1 component wave functions $\Psi_a = Q_a\Psi(a =$ 1,2,3). For this, it is sufficient to use the realization of $\gamma_a^{(1)}$ and $\gamma_a^{(2)}$ matrices presented in [7, p.168].

3. The above mentioned facts show that other symmetry operators for the Breit equation except mentioned in Theorems 1,2 in the considered classes do not exist. Here we present new integral of motion of the Breit equation (in the center-mass-system) which is a differential operator of the second order and can not be presented as a polynomial of operators (7). This integral of motion is

$$Q = \beta_{(1)}\beta_{(2)}[2(\bar{S}\cdot\bar{J})^2 - 2\bar{S}\cdot\bar{J} - \bar{J}^2], \tag{8}$$

where $\bar{J} = (J_1, J_2, J_3)$, $\bar{S} = (S_1, S_2, S_3)$, $J_a(a = 1, 2, 3)$ are operators (7), S_a is the matrix (5).

We can be convinced by the direct verification that operator (8) commutes with the Hamiltonian H (7) (and with the operator $L = i\dfrac{\partial}{\partial t} - H$), and therefore is the motion constant (symmetry operator). Formula (8) presents the motion constant also for a generalized Breit equation, when potentials V_1, V_2, V_3 from (3) are arbitrary functions of $\bar{x}, \beta_{(1)}, \beta_{(2)}$, and for the Bethe-Salpeter equation [1] (where interaction potential is calculated in the first approximation) and for the Barut-Komy equation [10].

We can show that in the space of square integrable functions the operator (8) spectrum is discrete and presented by the following formula:

$$Q\Psi = \varepsilon j(j+1)\Psi, \varepsilon \pm 1, \quad j = 0, 1, 2, \ldots \tag{9}$$

Motion constant (8) can be used for solution of the eigenvalue problem for the generalized Breit Hamiltonian.

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