Nonlinear Mathematical Physics 1996, V.3, N 3–4, 385–387.

Symmetry of a Two-Particle Equation for Parastates

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Abstract

We study hidden symmetry of a two-particle system of equations for parastates. Invariance operators are described for various potentials.

It is a well-known fact that the systems of partial differential equations have a hidden symmetry, which can not be observed in the classical approach of Lie [1].

This paper is devoted to investigation of the hidden symmetry of equations of the form:

$$\widehat{A}\overline{\Psi} \equiv (p_0 + [\beta_0, \beta_a]_- \; p^a - m\beta_c + V(x_c\overline{x}))\overline{\Psi} = 0, \tag{1}$$

where $\overline{\Psi}$ is a five-component complex function (column matrix)

$$\overline{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \end{pmatrix},$$

$$p_0 = i\frac{\partial}{\partial x_0}, \quad p^a = i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3, \quad \overline{x} = (x_1, x_2, x_3),$$

β_μ are the irreducible matrices of dimension 5×5, which satisfy the algebra:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\nu\lambda}\beta_\mu,$$

$$\mu, \nu, \lambda = 0, 1, 2, 3, \quad g_{00} = -g_{11} = -g_{22} = g_{33} = 1,$$

$V(x_0, \overline{x})$ is the interaction potential.

The matrices β_μ form is unessential. The possible realization of such matrices is

$$\beta_0 = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that the matrices

$$\begin{aligned} &\beta_\mu, \ S_{\mu\nu} = [\beta_\mu, \beta_\nu]_-, \quad R\beta_\mu = \tfrac{2}{3}(\beta_\nu \beta^\nu - \tfrac{5}{2} I)\beta_\mu, \\ &Z_{\mu\nu} = [\beta_\mu, \beta_\nu]_+ - 2g_{\mu\nu} I, \ I, \quad \mu, \nu = \overline{0,3} \end{aligned} \tag{2}$$

form the basis in the space of matrices of dimension 5×5. Here $[AB]_\pm = AB \pm BA$.

Let us designate the space of generalized functions by D' [2].

The potential $V(x_0, \overline{x})$ is a matrix function of dimension 5×5, the matrix elements of this function belong to D'. It can be represented in terms of the basis matrices (2):

$$\begin{aligned} V(x_0, \overline{x}) = F(x_0, \overline{x}) \cdot I + H^\mu(x_0, \overline{x})\beta_\mu + X^\mu(x_0, \overline{x}) R\beta_\mu + \\ V^{\mu\nu}(x_0, \overline{x}) S_{\mu\nu} + Y^{\mu\nu}(x_0, \overline{x}) Z_{\mu\nu} \end{aligned} \tag{3}$$

where the functions $F(x_0, \overline{x}) \in D', H^\mu(x_0, \overline{x}) \in D'$,

$$X^\mu(x_0, \overline{x}) \in D', \quad V^{\mu\nu}(x_0, \overline{x}) \in D', \quad Y^{\mu\nu}(x_0, \overline{x}) \in D',$$

$$V^{\mu\nu}(x_0, \overline{x}) = -V^{\nu\mu}(x_0, \overline{x}), \quad Y^{\mu\nu}(x_0, \overline{x}) = Y^{\nu\mu}(x_0, \overline{x}), \quad \mu, \nu = \overline{0,3}.$$

The equation (1) can be interpreted as a two-particle equation in the center-mass-system for spin 1/2 particles with equal masses.

The present investigation touches upon the determination of such sets of potentials $V(x_0, \overline{x}) \not\equiv 0$, for which equation (1) is invariant with respect to the found symmetry operators.

The case $V(x_0, \overline{x}) \equiv 0$ is investigated in [3].

Definition. The equation (1) is invariant with respect to the linear differential operator Q, if

$$[\widehat{A}, Q]_- = 0, \tag{4}$$

where

$$\widehat{A} = p_0 + S_{0a} p^a + m\beta_0 + V(x_0, \overline{x}). \tag{5}$$

Theorem. *The equation (1) is invariant with respect to the following operators:*

1. $\langle p_0 \rangle$, *if* $V = F(x_0, c) \cdot I$;
2. $\langle p_3 \rangle$, *if* $V = V(x_0, x_1, x_2, c)$;
3. $\langle p_0 + p_1 \rangle$, *if* $V = F(x_0, c) \cdot I$;
4. $\langle J_{12} \rangle$, *if* $V = F(d) \cdot I + H(d)\beta_0 +$

$$k(d)\beta_3 + X(d)R\beta_0 + A(d)R\beta_3 + T(d)(Z_{11} + Z_{22}) +$$

$$N(d)Z_{00} + Y(d)Z_{33} + G(d)Z_{03} + B(d)S_{03} + W(d)S_{12},$$

where $d = d(x_c, x_1^2 + x_2^2, x_3, c)$*;*

5. $\langle J_{23} - \frac{\varepsilon}{2}(p_0 + p_1)\rangle$, *if* $V = F(x_0, c)$;
6. $\langle J_{12} + \alpha p_3\rangle$, *if* $V = F(k)I + H(k)\beta_0 +$

$$K(k)\beta_3 + X(k)R\beta_0 + A(k)R\beta_3 + B(k)S_{03} + W(k)S_{12} +$$

$$G(k)Z_{03} + N(k)Z_{00} + T(k)(Z_{11} + Z_{22}) + Y(k)Z_{33},$$

where $k = k\left(x_0, x_1^2 + x_2^2, x_1 \cos\left(-\frac{x_3}{\alpha}\right) - x_2 \sin\left(-\frac{x_3}{\alpha}\right), c\right)$.

Here the functions $F, H, K, X, A, B, W, G, Y, N, T, V$ *belong to the space of generalized functions* D'*, functions* α *and* k *belong to the space* D *[2],* $\alpha \neq 0$*,* $\varepsilon = \pm 1$*,* $c = \text{const}$*,*

$$p_a = -i\frac{\partial}{\partial x_a}, \qquad p_0 = S_{0a}p_a + m\beta_0,$$

$$J_{ab} = x_a\rho_b,\ -x_b\rho_a + iS_{ab}, \quad a, b = 1, 2, 3.$$

The matrices $S_{\mu\nu}, R\beta_\mu, Z_{\mu\nu}, \mu\nu = \overline{0,3}$ *are assumed to satisfy relationships (2).*

Proof of the Theorem is rather cumbersome, that is why it would be better to give its algorithm:

1. Let us substitute the operator $\widehat{A}$ (5) into relationship (4), where $V(x_0, \overline{x})$ has the form (3), and one of symmery's operators Q_R, $R = \overline{1,6}$, according to the theorem. Then we simplify the received system of equations, using the relationships between the basis matrices.

2. We get the system of certain equations by equating the coefficients of the linear independent basis matrices (2) with the same operators of differentiation. We shall get the proof of the theorem by solving this system.

References

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