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Conditional and Lie Symmetry of Nonlinear Wave Equation

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Abstract

Group classification of the nonlinear wave equation is carried out and the conditional invariance of the wave equation with the nonlinearity $F(u) = u$ is found.

Let us consider the nonlinear wave equation

$$u_{00} + m\frac{u_0}{x_0} - p\frac{F(u)u_1}{x_1} - (F(u)u_1)_1 = 0, \tag{1}$$

where $u = u(x_0, x_1)$, $u_\mu = \dfrac{\partial u}{\partial x_\mu}$, $u_{\mu\mu} = \dfrac{\partial^2 u}{\partial x_\mu^2}$, $\mu = 0, 1$, $F(u)$ is an arbitrary differentiable function, $F'(u) \neq 0$, $F(u) > 0$, m and p are arbitrary constants.

Eq.(1) is widely used in mathematical physics. In the case $m = p = 0$ we have a well-known equation [1], namely

$$u_{00} - (F(u)u_1)_1 = 0. \tag{2}$$

Besides, one can obtain Eq.(1) when the symmetry reduction of the multidimensional wave equation

$$g_{\mu\nu}u_{\mu\nu} - (F(u)u_a)_a = 0,$$

where

$$g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu = 0, \\ -1, & \mu = \nu > 0, \end{cases} \qquad \mu = 0, \ldots, l, \quad a = l+1, \ldots, n$$

to a two-dimensional wave equation is carried out. Eq.(1) has the following property: the local substitution $\int F(u)\mathrm{d}u = v$ transforms Eq.(1) into the equation

$$v_{11} + p\frac{v_1}{x_1} - m\frac{\Phi(v)v_0}{x_0} - (\Phi(v)v_0)_0 = 0, \tag{3}$$

where $\Phi(v)$ is the function inverse to $\int F(u)\mathrm{d}u$.

Transpositions of x_0 and x_1, m and p in Eq.(3) lead to the equation from the class (1). This property facilitates investigation of the equation.

Results of symmetry classification of Eq.(1), which is made by the Lie approach [2], are given in Tables 1–4. It should be noted that the group properties of Eq.(2) were considered in detail in [1].

Case 1. $m = 0,\ p = 0$

Table 1

$F(u)$	Lie Algebra
arbitrary	$P_0 = \partial_0, P_1 = \partial_1, D_1 = x_0\partial_0 + x_1\partial_1$
e^u	$P_0, P_1, D_1, D_2 = x_1\partial_1 + 2\partial_u$
u^k	$P_0, P_1, D_1, D_3 = kx_1\partial_1 + 2u\partial_u$
u^{-4}	$P_0, P_1, D_1, D_3, K_1 = x_0^2\partial_0 + x_0 u\partial_u$
$u^{-4/3}$	$P_0, P_1, D_1, D_3, K_2 = x_1^2\partial_1 - 3x_1 u\partial_u$

Case 2. $m = 0,\ p \neq 0$

Table 2

$F(u)$	Lie Algebra
arbitrary	P_0, D_1
e^u	P_0, D_1, D_2
u^k	P_0, D_1, D_3
$u^{\frac{2(p-2)}{3-p}}$	$P_0, D_1, D_3, K_3 = x_1^{2-p}\partial_1 + (p-3)x_1^{1-p}u\partial_u$
u^{-4}	P_0, D_1, D_3, K_3, K_1

Case 3. $m \neq 0,\ p = 0$

Table 3

$F(u)$	Lie Algebra
arbitrary	P_1, D_1
e^u	P_1, D_1, D_2
u^k	P_1, D_1, D_3
$u^{\frac{2(m-2)}{1-m}}$	$P_1, D_1, D_3, K_4 = x_0^{2-m}\partial_0 + (1-m)x_0^{1-m}u\partial_u$
$u^{-4/3}$	P_1, D_1, D_3, K_4, K_2

Case 4. $m \neq 0,\ p \neq 0$

Table 4

$F(u)$	Lie Algebra
arbitrary	D_1
e^u	D_1, D_2
u^k	D_1, D_3
$u^{\frac{2(p-2)}{3-p}}$	D_1, D_3, K_3
$u^{\frac{2(m-2)}{3-m}}$	D_1, D_3, K_4
$u^{\frac{2(p-2)}{3-p}}$, $p + m = 4$	D_1, D_3, K_3, K_4

Results of the Q-conditional symmetry of Eq.(1) in the case $F(u) = u$ are adduced in the following theorems:

Theorem 1. *Equation*

$$u_{00} - (uu_1)_1 = 0$$

is Q-conditionally invariant under the following operators:

$$Q_1 = x_1 x_0^2 \partial_1 + \left(2x_0 + \lambda x_0^5\right) \partial_u, \quad \lambda = \mathrm{const},$$
$$Q_2 = 2x_1 x_0^2 \partial_1 + \left(u x_0^2 + 3x_1^2\right) \partial_u,$$
$$Q_3 = \partial_0 - 2x_0 \partial_1 + 8x_0 \partial_u,$$
$$Q_4 = x_0 \partial_0 - \left(6x_0^5 + x_1\right) \partial_1 + 2\left(u - 3\left(x_1^2 x_0^{-2} + 2x_1 x_0^3 - 24x_0^8\right)\right) \partial_u,$$
$$Q_5 = 2x_0 \partial_0 + \left(x_1 - 3x_0^2\right) \partial_1 - 2\left(u + 3x_1 - 9x_0^2\right) \partial_u,$$
$$Q_6 = x_0 \partial_0 - 3x_0^3 \partial_1 + \left(u + 27x_0^4\right) \partial_u.$$

Theorem 2. *Equation*

$$u_{00} + m\frac{u_0}{x_0} - (uu_1)_1 = 0$$

is Q-conditionally invariant under the following operators:

$$Q_7 = 2x_1 \partial_1 + \left(u + (3-m) x_1^2 x_0^{-2}\right) \partial_u,$$
$$Q_8 = \partial_0 - 2x_0 \partial_1 + 8x_0 \partial_u.$$

Theorem 3. *Equation*

$$u_{00} - p\frac{uu_1}{x_1} - (uu_1)_1 = 0$$

is Q-conditionally invariant under the following operator:

$$Q_9 = (p+3) x_0^2 \partial_1 + 6x_1 \partial_u.$$

The algorithm of the Q-conditional symmetry is given in [3]. It should be noted that the conditional symmetry of Eq.(2) for different $F(u)$ is studied in [4].

References

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