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Generalized Self-Duality for the Supersymmetric Yang-Mills Theory with a Scalar Multiplet

V.A. YATSUN and A.M. PAVLYUK

Bogolyubov Institute for Theoretical Physics of the National Ukrainian Academy of Sciences, 14b Metrologichna Street, Kyïv 143, 252143, Ukraïna

Abstract

Generalized self-duality equations for the supersymmetric Yang-Mills theory with a scalar multiplet are presented in terms of component fields and superfields as well.

1 Generalized self-duality equations in component fields

In the ordinary non-supersymmetric Yang-Mills theory the self-duality equations read

$$F_{mn} =^* F_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F^{kl}, \tag{1}$$

where

$$F_{mn} = \partial_m V_n - \partial_n V_m + ig\,[\,V_m\,,\,V_n\,].$$

Going from vector indices to spinor indices: $x_m \to x_{\alpha\dot\alpha} = \sigma^m{}_{\alpha\dot\alpha}x_m$, one obtains the Yang-Mills strength in the following form

$$F_{\alpha\dot\alpha,\beta\dot\beta} \equiv \sigma^m{}_{\alpha\dot\alpha}\sigma^n{}_{\beta\dot\beta}F_{mn} = \frac{1}{2}\varepsilon_{\alpha\beta}f_{\dot\alpha\dot\beta} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}f_{\alpha\beta}\,,$$

where

$$f_{\dot\alpha\dot\beta} \equiv F^{\alpha}{}_{\dot\alpha,\alpha\dot\beta}\,, \qquad f_{\alpha\beta} \equiv F_{\alpha}{}^{\dot\beta}{}_{,\beta\dot\beta}\,.$$

Now the self-duality equations (1) take the form

$$f_{\alpha\beta} = 0. \tag{2}$$

In the supersymmetric Yang-Mills theory the equations (2) must be supplemented by equations for superpartners and auxiliary field. So one obtains the following system of super self-duality equations in terms of the component fields belonging to a vector multiplet [4]

$$\begin{aligned} &f_{\alpha\beta} = 0,\\ &\mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} = 0,\\ &\lambda_\alpha = 0, \quad D = 0, \end{aligned} \tag{3}$$

which is invariant under supersymmetric transformations.

In this report we consider the supersymmetric Yang-Mills theory with a scalar multiplet. It is described by two multiplets of the component fields: the vector multiplet

$$V = (V_m\,;\,\lambda\,;\,D)\,,$$

and scalar one

$$\Phi = (A\,,\,B\,;\,\psi\,;\,F\,,\,G)\,.$$

A gauge-invariant supersymmetric Lagrangian of the theory, whose fields belong to the adjoint representation of a gauge group, has the form [7]

$$\begin{gathered} L = \mathrm{Tr}\Big\{-\frac{1}{4}F_{mn}^2 + \frac{i}{2}\bar\lambda\gamma^m\mathcal{D}_m\lambda - \frac{1}{2}(\mathcal{D}_mA)^2 - \frac{1}{2}(\mathcal{D}_mB)^2 + \frac{i}{2}\bar\psi\gamma^m\mathcal{D}_m\psi + \\ ig\bar\lambda\,[\,A+\gamma_5B\,,\,\psi\,] + igD\,[\,A\,,\,B\,] + \frac{1}{2}D^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2\Big\}\,, \end{gathered} \tag{4}$$

where

$$F_{mn} = \partial_mV_n - \partial_nV_m + ig\,[\,V_m\,,\,V_n\,]\,,$$

$$\mathcal{D}_mA = \partial_mA + ig\,[\,V_m\,,\,A\,]\,.$$

A 4-component Majorana spinor λ consists of 2-component Weyl spinors

$$\lambda = \begin{pmatrix}\lambda_\alpha\\ \bar\lambda^{\dot\alpha}\end{pmatrix},$$

and its conjugate is $\bar\lambda = \lambda^+\gamma^0 = -(\lambda^\alpha,\,\bar\lambda_{\dot\alpha})$. Similarly, for a Majorana spinor ψ. Here γ-matrices are taken in the Weyl representation

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar\sigma^m & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

The equations of motion look as follows (in 2-spinor notation)

$$\begin{aligned}
&\mathcal{D}_\alpha{}^{\dot\alpha} f_{\dot\alpha\dot\beta} + \mathcal{D}^\beta{}_{\dot\beta} f_{\alpha\beta} + 8g(\{\lambda_\alpha\,,\,\bar\lambda_{\dot\beta}\} + \{\psi_\alpha\,,\,\bar\psi_{\dot\beta}\}) - \\
&\qquad 2ig([\,A-iB\,,\,\mathcal{D}_{\alpha\dot\beta}(A+iB)\,] + [\,A+iB\,,\,\mathcal{D}_{\alpha\dot\beta}(A-iB)\,]) = 0,\\
&\mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}(A-iB) - 4ig\{\,\lambda^\alpha,\ \psi_\alpha\,\} - 2g\,[\,D\,,\,A-iB\,] = 0,\\
&\mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}(A+iB) - 4ig\{\,\bar\lambda_{\dot\alpha}\,,\,\bar\psi^{\dot\alpha}\,\} + 2g\,[\,D\,,\,A+iB\,] = 0,\\
&D + ig\,[\,A\,,\,B\,] = 0,\\
&F = G = 0,\\
&\mathcal{D}^{\alpha\dot\beta}\lambda_\alpha - g\,[\,\bar\psi^{\dot\beta}\,,\,A-iB\,] = 0,\\
&\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} - g\,[\,\psi_\alpha\,,\,A+iB\,] = 0,\\
&\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g\,[\,\bar\lambda^{\dot\beta}\,,\,A-iB\,] = 0,\\
&\mathcal{D}_{\alpha\dot\beta}\bar\psi^{\dot\beta} + g\,[\,\lambda_\alpha\,,\,A+iB\,] = 0\,.
\end{aligned} \tag{5}$$

These equations of motion are covariant under usual supersymmetric transformations:

$$
\begin{aligned}
&\delta_\xi(A - iB) = 2\xi\psi,\\
&\delta_\xi V_m = i\xi\sigma_m\bar{\lambda} + i\bar{\xi}\bar{\sigma}_m\lambda,\\
&\delta_\xi\lambda_\alpha = -\tfrac{1}{2}\xi^\beta f_{\alpha\beta} + i\xi_\alpha D,\\
&\delta_\xi\psi_\alpha = i\bar{\xi}^{\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}(A - iB) + \xi_\alpha(F + iG),\\
&\delta_\xi D = -\xi\sigma^m\mathcal{D}_m\bar{\lambda} + \bar{\xi}\bar{\sigma}^m\mathcal{D}_m\lambda,\\
&\delta_\xi(F + iG) = 2i\bar{\xi}_{\dot{\beta}}(\mathcal{D}^{\alpha\dot{\beta}}\psi_\alpha + g\,[\,\bar{\lambda}^{\dot{\beta}}\,,\,A - iB\,])\,,
\end{aligned}
\tag{6}
$$

where $\xi_\alpha\,,\,\bar{\xi}_{\dot{\alpha}}$ are the parameters of the supersymmetry transformations.

The equations of motion (5) are the second-order equations. Here we postulate the system of first-order equations, which lead to the equations of motion (5)

$$
\begin{aligned}
&f_{\alpha\beta} = 2gc_{\alpha\beta}[\,A\,,\,B\,],\\
&(\mathcal{D}_{1\dot{\alpha}} - k_1\mathcal{D}_{2\dot{\alpha}})(A - iB) = 0,\\
&(\mathcal{D}_{1\dot{\alpha}} - k_2\mathcal{D}_{2\dot{\alpha}})(A + iB) = 0,\\
&D + ig\,[\,A\,,\,B\,] = 0,\\
&F = G = 0,\\
&\mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} - g\,[\,\psi_\alpha\,,\,A + iB\,] = 0,\\
&\mathcal{D}^{\alpha\dot{\beta}}\psi_\alpha + g\,[\,\bar{\lambda}^{\dot{\beta}}\,,\,A - iB\,] = 0,\\
&\lambda_\alpha = \bar{\psi}_{\dot{\alpha}} = 0,\\
&\psi_1 = k_1\psi_2\,,
\end{aligned}
\tag{7}
$$

where the coefficients $c_{\alpha\beta}\,,\,k_1\,,\,k_2$ are complex numbers and $c_{\alpha\beta}$ is symmetric in $\alpha\,,\,\beta$. The coefficients are connected by the relations

$$
\begin{aligned}
&c_{12} - k_1c_{22} = -1\,, &\qquad& c_{12} - k_2c_{22} = 1,\\
&c_{11} - k_1c_{12} = -k_1\,, && c_{11} - k_2c_{22} = k_2\,.
\end{aligned}
\tag{8}
$$

It is easy to see that among five coefficients only two of them are independent. The system (7) is covariant under usual supersymmetric transformations (6) if there exists the following relation between the parameters of supersymmetry

$$\xi_1 = k_2\xi_2. \tag{9}$$

We call the system (7) *generalized self-duality equations.* Generalization means that in absence of a scalar multiplet we obtain super self-duality equations in component fields for the supersymmetric Yang-Mills theory.

2 Generalized self-duality equations in superfield formulation

A superfield formulation of the generalized self-duality equations (7) can be obtained in the superspace $(x^m\,;\,\theta_\alpha\,;\,\bar\theta_{\dot\alpha})$ with the fixed Grassmanian variable $\theta_1 = k_2\,\theta_2$. A spinor chiral superfield is defined by

$$W_\alpha = -\frac{1}{8g}\overline{\mathcal{D}\mathcal{D}}(e^{-2gV}\mathcal{D}_\alpha e^{2gV})\,,$$

where V is a vector superfield:

$$V = -\theta^\alpha\bar\theta^{\dot\alpha}V_{\alpha\dot\alpha} + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D$$

and the covariant derivatives are

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\,, \qquad \overline{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\partial_{\alpha\dot\alpha}\,.$$

A scalar chiral superfield is defined by the condition $\overline{\mathcal{D}}_{\dot\alpha}\Phi = 0$ and has the following form in components

$$\Phi = \frac{1}{2}(A-iB)+\theta\psi+\frac{i}{2}\theta^\alpha\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}(A-iB)+\frac{i}{2}\theta\theta\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^\alpha+\frac{1}{8}\theta\theta\bar\theta\bar\theta\,\Box(A-iB)+\frac{1}{2}\theta\theta(F+iG)\,.$$

Consequently its Hermite conjugate Φ^+ is given by $\mathcal{D}_\alpha\Phi^+ = 0$. The gauge-covariant spinor derivatives are

$$\nabla_\alpha = \mathcal{D}_\alpha + i\,[\,\mathcal{A}_\alpha\,,\,\cdot\,]\,, \qquad \nabla_{\dot\alpha} = \overline{\mathcal{D}}_{\dot\alpha} + i\,[\,\mathcal{A}_{\dot\alpha}\,,\,\cdot\,]\,,$$

where

$$\mathcal{A}_\alpha = -ie^{-2gV}\mathcal{D}_\alpha e^{2gV}\,, \qquad \mathcal{A}_{\dot\alpha} = i\,\overline{\mathcal{D}}_{\dot\alpha}e^{2gV}\cdot e^{-2gV}\,.$$

The system of the generalized self-duality equations (7) can be written in terms of superfields in the sub-superspace $\theta_1 = k_2\theta_2$ in the following form

$$\begin{aligned}
&\nabla_\alpha W_\beta = 2g\rho_{\alpha\beta}\,[\,\Phi\,,\,\Phi^+ - 2g\,[\,V\,,\,\Phi^+\,]],\\
&\nabla_1\Phi = k_1\nabla_2\Phi,\\
&\nabla_{\dot\alpha}\Phi^+ = 0\,,
\end{aligned} \tag{10}$$

where the coefficients $\rho_{\alpha\beta}$ are expressed through $c_{\alpha\beta}$ and can be organized into the following matrix

$$\rho = -\begin{pmatrix} c_{11} & -1+c_{12} \\ 1+c_{12} & c_{22} \end{pmatrix}$$

with zero determinant.

References

[1] Semikhatov A.M., *Phys. Lett. B,* 1983, V.120, 171.

[2] Grimm R., Sohnius M. and Wess J., *Nucl. Phys. B,* 1978, V.133, 275.

[3] Sohnius M., *Nucl. Phys. B,* 1978, V.136, 461.

[4] Volovich I.V., *Phys. Lett. B,* 1983, V.123, 329.

[5] Devchand Ch. and Ogievetsky V., The matreoshka of supersymmetric self-dual theories, BONN-HE-93-23.

[6] Devchand Ch. and Ogievetsky V., The super self-dual matreoshka, BONN-HE-93-33.

[7] Ferrara S. and Zumino B., *Nucl. Phys. B,* 1974, V.79, 413.

[8] Yatsun V.A., *Ukr. Fiz. Zhurn.,* 1991, V.36, 1828.

[9] Yatsun V.A., Quasi-selfduality and supersymmetry breaking, to be published.

[10] Wess J. and Bagger J., Supersymmetry and Supergravity, Princeton University Press, 1983.