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Symmetry Reductions of the Lax Pair of the Four-Dimensional Euclidean Self-Dual Yang-Mills Equations

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Abstract

The reduction by symmetry of the linear system of the self-dual Yang-Mills equations in four-dimensions under representatives of the conjugacy classes of subgroups of the connected part to the identity of the corresponding Euclidean group under itself is carried out. Only subgroups leading to systems of differential equations nonequivalent to conditions of zero curvature without parameter, or to systems of uncoupled first order linear O.D.E.'s are considered. Lax pairs for a modified form of the Nahm's equations as well as for systems of partial differential equations in two and three dimensions are written out.

1 Introduction

Several systems of partial differential equations have been investigated in the past via the method of symmetry reduction (see Refs. [1–3], and references therein). This includes the (coupled) Yang-Mills theories [4–9], and in particular the self-dual Yang-Mills (abbreviated SDYM below) equations in flat spaces. In four dimensions, the latter equations are known to be completely solvable through the twistor construction [10, 11]. Their reductions under symmetries, with often the addition of algebraic constraints, have produced a large number of known integrable systems in lower dimensions (for details, see Refs. [12] and [13]), such as: the Nahm [14–16], Boussinesq [17–19], (modified) Korteweg-de Vries [17–19, 20, 21], (generalized) nonlinear Schrödinger [19–21], N-wave [22] and Kadomtsev–Petviashvili [22] equations. Most of these reductions have been accomplished using only translations, and their hierarchies have been examined through the same reductions in Ref. [23]. Moreover, symmetry reductions using different invariant fields, or ansatze, have been effected for higher dimensional versions of the SDYM equations [24–26] as well as for some generalizations to self-dual spaces [27].

The corresponding linear system, or Lax pair [10,11,28,29], to the SDYM equations has also been reduced with respect to translations as well as other two- and three-dimensional Abelian subgroups of the conformal group [30–32]. As expected, the compatibility of the reduced Lax pair led to the SDYM equations reduced under the same symmetry group. Let us mention that the six trancendents of Painlevé were found in Ref. [32] as the result of reductions with respect to three-dimensional Abelian subgroups, which have also been derived through further reductions of reduced systems of the SDYM equations (see Ref. [13]). The symmetries involved consisted of translations, rotations, and dilatations. In particular, the nontrivial reduction of the Lax pair for the SDYM equations to the Lax pair for the Painlevé equation P_{VI} has been exhibited.

In the following, the symmetry reduction of the Lax pair of the SDYM equations with respect to any subgroup of the conformal group is described. In our discussion, we will restrict ourselves to the Euclidean version of the SDYM system, where preliminary results have been obtained [33]. This work can also be performed on R^4 endowed with the diagonal metric: $(+1,+1,-1,-1)$ ($R^{(2,2)}$) [34, 35]. Let us recall that the procedure of symmetry reduction has been applied to generate new gauge invariant or supersymmetric systems from higher dimensional ones. There is in general a residual gauge symmetry after reduction, but no residual supersymmetry is ensured. Despite this result, supersymmetric extensions of the SDYM equations in Euclidean space have been reduced by subgroups of the Euclidean group (Ref. [36] and references therein) and supersymmetric versions of known integrable systems have been produced. Similarly, some superintegrable systems, such as the super-Korteweg-de Vries and super-Toda field equations, have been recovered from the supersymmetric SDYM equations in $R^{(2,2)}$ with the help of differential constraints [37]. As a further motivation to this work, let us mention that extended self-dual supersymmetric Yang-Mills theories correspond to low energy limits of open or heterotic $N = 2$ superstring theories [38, 39].

In order to introduce our notation, we recall in section 2 the four-dimensional SDYM equations as well as their corresponding linear system, or Lax pair, in Euclidean space (E^4). Then, the lift of the action of the conformal group $SO(5,1)$, which leaves invariant both the SDYM equations and its Lax pair is found by reference to the twistor construction. Section 3 reviews the invariance conditions for the different elements involved in the linear system: i.e. the Yang-Mills fields and the multiplet of scalar fields (vector-functions), transforming under the fundamental representation of the gauge group. A classification of the subalgebras of the real Euclidean Lie algebra, $e(4) \sim so(4) \triangleright t^4$, of the Euclidean group ($E(4) \sim O(4) \circledS T^4$), with respect to its connected part to the identity ($E_o(4)$) is also indicated. We then describe an algorithm of symmetry reduction of the Lax pair for the SDYM equations and provide explicit examples in section 4. The reduced Lax pairs obtained through reductions under representatives of the conjugacy classes of subgroup of $E_o(4)$ giving rise to nontrivial nonlinear differential systems of reduced equations are presented in section 5. We end this article with a summary of the results and some comments regarding future directions of this work.

2 Self-Dual Yang-Mills Equations and Lax Pair

Let us write the SDYM equations in E^4 to set our notation:

$$F = *F, \tag{2.1}$$

where F is a curvature 2-form pulled back to E^4 from the gauge bundle $P(E^4, H)$, explicitly : $F = d\omega + \omega \wedge \omega$, with the connection 1-form ω on P taking values in the Lie algebra $\mathcal{H}$ of the gauge group H.

In terms of Cartesian coordinates $\{x^\mu\}$, they can be expressed as:

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\sigma}F_{\kappa\sigma}, \tag{2.2}$$

where $\mu, \nu, ... = 1, ..., 4$, $\epsilon_{\mu\nu\kappa\sigma}$ stands for the completely antisymmetric tensor in four dimensions with the convention: $\epsilon_{1234} = 1$. The components of the field strength $(F_{\mu\nu})$ are given by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2.3}$$

The solutions to the SDYM equations on complexified Minkowski space (M^C), or self-dual connections on a vector bundle N over M^C, are related in a one-to-one manner to holomorphic vector bundles $\tilde{N}$ over CP^{3*}, trivial over CP^1 submanifolds. The fibres (C^n) of the bundle $\tilde{N}$ consist of covariantly constant sections Ψ of the bundle N on anti-self-dual planes of M^C corresponding to a point of CP^{3*}, which are also called β-planes. The condition of self-duality, or the self-dual equations, of the connections on M^C is in fact equivalent to the condition of covariant constancy of sections Ψ with respect to the (self-dual) connection on the anti-self-dual planes in M^C. The latter condition can be interpreted as a Lax pair. By imposing a suitable antiholomorphic involution on CP^{3*}, a fibration $CP^{3*} \to S^4$ with "real lines" CP^1 is induced, hence $CP^{3*} = CP^{3*}(S^4, CP^1)$. In the same manner, the vector bundle over S^4 is pulled back to a holomorphic vector bundle over CP^{3*} if the self-duality condition on S^4 is satisfied [10, 11, 40]. These constraints correspond to a Lax pair and can be expressed as follows if we introduce a chart R^4 of S^4 with coordinates x^μ:

$$[D_1 + iD_2 - \lambda(D_3 + iD_4)]\Psi(x, \lambda, \bar{\lambda}) = 0, \tag{2.4a}$$

$$[D_3 - iD_4 + \lambda(D_1 - iD_2)]\Psi(x, \lambda, \bar{\lambda}) = 0, \tag{2.4b}$$

$$\partial_{\bar{\lambda}}\Psi(x, \lambda, \bar{\lambda}) = 0, \tag{2.4c}$$

where the covariant derivative: $D_\mu := \partial_\mu + A_\mu$ and $\lambda \in CP^1$. The vector-function, or multiplet of scalar fields, Ψ is a holomorphic section of the vector bundle $\tilde{N}$ over CP^{3*}, as expressed by (2.4c).

On the subset $R^4 \times R^2$ of CP^{3*}, labelled by the coordinates $(x^\mu, y^i), i = 1, 2$, one finds that the vector parts of eqs (2.4):

$$\partial_1 + i\partial_2 - \lambda\partial_3 - i\lambda\partial_4, \tag{2.5a}$$

$$\lambda\partial_1 - i\lambda\partial_2 + \partial_3 - i\partial_4, \tag{2.5b}$$

and

$$\partial_{\bar{\lambda}} = \frac{1}{2}\left(\frac{\partial}{\partial y^1} + i\frac{\partial}{\partial y^2}\right), \tag{2.5c}$$

define a basis of antiholomorphic vector fields with respect to the complex structures $\mathcal{J}$ on CP^{3*}:

$$\mathcal{J} = \{J^\nu_\mu = -s_a\delta^{\nu\rho}\bar{\eta}^a_{\rho\mu},\ \epsilon^j_i\}, \tag{2.6}$$

where the antisymmetric two-index tensor ϵ^j_i $(i, j, ... = 1, 2)$ is normalized to unity $(\epsilon^2_1 = -1)$, s_a $(a, b, ... = 1, 2, 3)$ are the Cartesian coordinates on R^3 such that $s_a s_a = 1$, which parametrize the fibre $CP^1 \simeq S^2$, and a chart of S^2 with variables (y^1, y^2) where: $\lambda = y^1 + iy^2$, has been chosen via a stereographic projection. Explicitly, we have:

$$s_1 = \frac{2y^1}{1+|y|^2} = \frac{\lambda+\bar{\lambda}}{1+\lambda\bar{\lambda}},$$

$$s_2 = \frac{2y^2}{1+|y|^2} = \frac{i(\bar{\lambda}-\lambda)}{1+\lambda\bar{\lambda}},$$

$$s_3 = \frac{1-|y|^2}{1+|y|^2} = \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}, \tag{2.7}$$

where $|y|^2 = (y^1)^2 + (y^2)^2$. The 't Hooft tensor $(\eta^a_{\mu\nu})$ and its dual $(\bar{\eta}^a_{\mu\nu})$ are given by (see Ref. [41] for identities):

$$\eta^a_{bc} = \bar{\eta}^a_{bc} = \epsilon_{abc}, \tag{2.8a}$$

where $a, b, c = 1, 2, 3$ and ϵ_{abc} is the three-dimensional antisymmetric tensor $(\epsilon_{123} = 1)$,

$$\bar{\eta}^a_{b4} = -\eta^a_{b4} = -\delta^a_b, \tag{2.8b}$$

$$\eta^a_{\mu\nu} = -\eta^a_{\nu\mu}, \tag{2.8c}$$

$$\bar{\eta}^a_{\mu\nu} = -\bar{\eta}^a_{\nu\mu}. \tag{2.8d}$$

The SDYM equations (2.2) and their linear system (2.4) are invariant under the gauge transformations:

$$A'_\mu = h^{-1}A_\mu h + h^{-1}\partial_\mu h, \tag{2.9a}$$

and

$$\Psi' = h^{-1}\Psi, \tag{2.9b}$$

where $h \in H$ is a function of $x \in S^4$. These equations are also preserved by the global action of the conformal group $SO(5,1)$. In order to preserve the holomorphic structure of the bundle $\tilde{N} \to CP^{3*}$, the action of $SO(5,1)$ is lifted to CP^{3*} in a holomorphic fashion by requiring the complex structure (2.6) to be invariant with respect to a lifted action of the conformal group. Locally, the lifted vector fields $(\tilde{X})$ will obey to the Lie algebra $so(5,1)$ of $SO(5,1)$, and will correspond to infinitesimal automorphisms of the complex structure (2.6), i.e. [42, 43]:

$$\mathcal{L}_{\tilde{X}}\mathcal{J} = 0, \tag{2.10}$$

$\forall X \in so(5,1)$, where $\mathcal{L}_{\tilde{X}}$ denotes the Lie derivative with respect to $\tilde{X}$.

A specific representation of $so(5,1)$ can be realized in terms of vector fields $(\hat{X})$ on E^4

$$\hat{X}_a = -\frac{1}{2}\delta_{ab}\eta^b_{\mu\nu}x_\mu\partial_\nu, \quad \hat{Y}_a = -\frac{1}{2}\delta_{ab}\bar{\eta}^b_{\mu\nu}x_\mu\partial_\nu, \quad \hat{P}_\mu = \partial_\mu,$$

$$\hat{K}_\mu = \frac{1}{2}x_\sigma x_\sigma\partial_\mu - x_\mu\hat{D}, \quad \hat{D} = x_\sigma\partial_\sigma, \tag{2.11}$$

where $\{\hat{X}_a, a = 1, 2, 3\}$ and $\{\hat{Y}_a, a = 1, 2, 3\}$ are two commuting $so(3)$ Lie algebras of $so(4)$, $\hat{K}_\mu$ denotes the generators of the special conformal transformations, and $\hat{D}$ generates the dilatations.

One verifies that the lifted vector fields on $R^4 \times CP^1 = CP^{3*} \backslash CP^1$ can be expressed as:

$$\tilde{X}_a = \hat{X}_a, \quad \tilde{Y}_a = \hat{Y}_a - Z_a, \quad \tilde{P}_\mu = \hat{P}_\mu,$$
$$\tilde{K}_\mu = \hat{K}_\mu + \bar{\eta}^a_{\sigma\mu} x_\sigma Z_a, \quad \tilde{D} = \hat{D}, \tag{2.12}$$

with the generators $\{Z_a, a = 1,2,3\}$ forming the $SO(3)$ rotations on S^2, or vector fields $\in T(CP^1)$ such that:

$$Z_a = \epsilon_{abc} s_b \partial_c, \tag{2.13}$$

which in terms of the parameter λ become:

$$Z_1 = \frac{i}{2}[(\lambda^2 - 1)\partial_\lambda + (1 - \bar{\lambda}^2)\partial_{\bar{\lambda}}],$$
$$Z_2 = \frac{1}{2}[(1 + \lambda^2)\partial_\lambda + (1 + \bar{\lambda}^2)\partial_{\bar{\lambda}}],$$
$$Z_3 = i(\lambda\partial_\lambda - \bar{\lambda}\partial_{\bar{\lambda}}). \tag{2.14}$$

Let us restrict ourselves to the (real) Euclidean Lie algebra $e(4) \sim so(4) \triangleright t^4$, which can be realized as an embedding in $so(5,1)$. We introduce the matrix $I_{5,1}$, defined as:

$$I_{5,1} = \left[\begin{array}{c|cc} \mathbf{1}_4 & 0 & \\ \hline & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]. \tag{2.15}$$

Then $so(5,1)$ consists of the set of elements $S \in gl(6,R)$ satisfying the relation:

$$S^T I_{5,1} + I_{5,1} S = 0. \tag{2.16}$$

Among those elements, the algebra $e(4)$ is determined by the subset composed of:

$$\left[\begin{array}{c|c} \mathbf{Y} & 0 \\ \hline 0 & \mathbf{0}_2 \end{array}\right], \quad \left[\begin{array}{cccc|cc} & & & & \alpha_1 & \alpha_1 \\ & & \mathbf{0}_4 & & \alpha_2 & \alpha_2 \\ & & & & \alpha_3 & \alpha_3 \\ & & & & \alpha_4 & \alpha_4 \\ \hline \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & 0 \end{array}\right], \tag{2.17}$$

where $Y^T = -Y$, each element belonging to $so(4) \sim so(3) \oplus so(3)$, and the translations (P_μ) along the x^μ-axis, parametrized by α_μ.

The linear action of the Euclidean group on R^6 provided with the Cartesian coordinates $(\eta^1, ..., \eta^6)$ determines the standard action of $E_o(4)$ on E^4 through the formula:

$$x^\mu = \frac{\eta^\mu}{\eta^5 + \eta^6}. \tag{2.18}$$

We have elected the following basis† of $so(4) \subset gl(6, R)$:

$$A_1 = -X_3 = \frac{1}{2}(M_{12} + M_{34}),\ A_2 = -X_1 = \frac{1}{2}(M_{23} + M_{14}),\ A_3 = X_2 = \frac{1}{2}(M_{13} - M_{24}), \tag{2.19}$$

$$B_1 = -Y_3 = \frac{1}{2}(M_{12} - M_{34}),\ B_2 = -Y_1 = \frac{1}{2}(M_{23} - M_{14}),\ B_3 = Y_2 = \frac{1}{2}(M_{13} + M_{24}), \tag{2.20}$$

$$[M_{\alpha\beta}]_{\mu\nu} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}, \quad [M_{\alpha\beta}]_{56} = 0, \tag{2.21}$$

where $\alpha, \beta, \mu, \nu = 1, .., 4$. Let us note that $M_{\alpha\beta}$ generate rotations in the (x^α, x^β)-plane.

Since the SDYM equations and their Lax pair are left unchanged by the action of $E_o(4)$, reductions with respect to subgroups conjugated under $E_o(4)$ will produced equivalent reduced systems (see Refs. [1–3]). We can therefore limit ourselves to reductions by a subgroup representative of each conjugacy class of subgroups of $E_o(4)$ [44]. Such a classification has been carried out for different subalgebras of interest in physics (see Refs. [3, 44–47] and references therein) and a "normalized" list of representatives of the conjugacy classes of subalgebras of $e(4)$ under $E_o(4)$ has been obtained in Ref. [44].

3 Invariance Conditions

The linear system (2.4) of the SDYM equations involves Yang-Mills fields (A_μ) and multiplets (Ψ) of scalar fields transforming under the fundamental representation of the gauge group H.

The Yang-Mills fields can be interpreted as pullbacks to the base manifold of connection 1-forms on $P(E^4, H)$, and their invariance has been studied in many papers. For instance, one may consult Refs. [5–8].

All the isotropy subgroups of the representatives of the conjugacy classes with orbits of dimension one, two, or three correspond either to the identity or to a compact Lie group: $SO(2)$ or $SO(3)$, and the approach presented in Refs. [5, 6] and [8] can then be used to determine the most general and globally invariant gauge fields in E^4. However, we are only interested in local expressions for the symmetric fields, and we will impose the infinitesimal form of these conditions.

Let us suppose that the symmetry group G acts (effectively) on each orbit G/G_o with cross-section 5, where G_o is identified as the isotropy subgroup of G at each point of 5.

In finite form, the invariance conditions are given by [5, 6, 8]:

$$f_g^* \omega = \rho^{-1}(g, x)\, \omega\, \rho(g, x) + \rho^{-1}(g, x) d\, \rho(g, x), \tag{3.1}$$

where $\omega^\sigma = A_\mu\, dx^\mu$.

†The notation A_1, A_2, A_3, B_1, B_2, and B_3 was previously used in ref. [44].

Infinitesimally, we have that:

$$\mathcal{L}_{\tilde{X}}\omega^\sigma = DW := dW + [\omega^\sigma, W], \tag{3.2}$$

$\forall g \in G$, or $\forall X \in \mathcal{G}$ (the Lie algebra of G), where the gauge transformation $\rho : G \times E^4 \to H$, specifies the lift of the group action (f_g) to the gauge bundle over $G/G_o \times V$ (cf Refs [5, 6] and [8]). The latter equation can be interpreted as the vanishing of the Lie derivative of the Yang-Mills fields with respect to a vector field $\tilde{X}$ induced by an element X of the symmetry algebra $\mathcal{G}$ up to an infinitesimal gauge transformation, where $W : \mathcal{G} \times E^4 \to \mathcal{H}$(the Lie algebra of H). Let us point out that the map W is defined as:

$$W = \frac{d}{dt}\rho(g = e^{tX}, x)|_{t=0}, \tag{3.3}$$

Its vanishing leads to a strict invariance condition, i.e. an invariance of the field without the help of any gauge transformation.

As for the multiplet of scalar fields, their finite and infinitesimal invariance conditions can be respectively read as:

$$\tilde{f}_g^* \Psi = \rho^{-1}(g, x)\, \Psi, \tag{3.4}$$

and

$$\mathcal{L}_{\tilde{X}^*}\Psi = -W\Psi, \tag{3.5}$$

$\forall X \in \mathcal{G}$, where $\tilde{f}_g^*$ and $\tilde{X}^*$ are respectively the lifts of the action and of the vector field associated to X.

For simplicity, all the cases below involve only an invariance without gauge transformation. Other reduced systems might be derived by substitution in the SDYM equations of invariant fields solutions to (3.1), (3.4) or (3.2), (3.5) with non-vanishing ρ and W functions (cf Ref. [33]).

4 Reduction by Symmetry of the Lax Pair

Let us consider a symmetry group G, subgroup of the invariance group $SO(5,1)$. The procedure of symmetry reduction consists essentially in substituting G-invariant A_μ and Ψ on $S^4 \times CP^1$ in the set of differential equations, rewritten in terms of the orbit and invariant coordinates of the G-action. Once a basis of a n-dimensional representative $\mathcal{G}$ of a conjugacy class of the subalgebras of $e(4)$ is chosen: $\{X_i, i = 1, ..., n\}$, we first determine its induced vector fields on $R^4 \times CP^1 \subset S^4 \times CP^1$: $\{\tilde{X}_i, i = 1, ..., n\}$, then select orbit variable(s): $\{\xi_m, m = 1, ..., \leq n\}$, normally group parameters, and determine the invariant coordinates: $\{\chi_A\}$, with the formula:

$$\mathcal{L}_{\tilde{X}_i}\chi_A = 0, \tag{4.1}$$

$\forall i = 1, ..., n$. Among the invariant variables χ_A, we can identify a (new) spectral parameter, denoted ζ, which obeys to $\mathcal{L}_{\hat{X}_i}\zeta \neq 0$, for some $i = 1, ..., n$.

After insertion in the Lax pair (2.4) of the coordinates (ξ_m, χ_A) and the G-invariant fields A_μ and Ψ, obtained via the method presented in the previous section, we are left using the holomorphicity condition (2.4c) and by elimination of multiplying factors functions of the orbits coordinates, with two reduced equations depending solely on invariant variables.

We can rewrite these reduced equations by dividing their differential and potential parts as:

$$\nabla_X \Psi = (X + A_X)\Psi = 0, \tag{4.2a}$$

and

$$\nabla_Y \Psi = (Y + A_Y)\Psi = 0, \tag{4.2b}$$

where X and Y represent the respective vector components of the reduced equations of the Lax pair (2.4).

It can be verified that the compatibility of the system (4.2) coincide with the SDYM equations reduced under the same subgroup G:

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = 0. \tag{4.3}$$

Let us add for a simple treatment that the equation (2.4c) can be interpreted as an invariance of A_μ (trivially) and Ψ under the translations ($P_{\bar{\lambda}}$) along the complex coordinate $\bar{\lambda}$ on $R^2 \subset CP^1$.

However, the compatibility (4.3) of (4.2) does not necessarily tally with the reduced SDYM equations if the residual vector components span only a one dimensional vector space. Still, the equations (4.2) respect the equality:

$$[\nabla_X, \nabla_Y]\Psi = f_X \nabla_X \Psi + f_Y \nabla_Y \Psi, \tag{4.4}$$

where f_X and f_Y are functions of the invariant coordinate(s), which give rise to the residual SDYM equations when appropriately chosen. A reduced Lax pair, producing the correct SDYM equations after reduction with respect to the same symmetry group is obtained if certain multiplying factors are adjoined to each of the operators ∇_X and ∇_Y.

In fact, we have:

$$[h_X \nabla_X, h_Y \nabla_Y] = 0, \tag{4.5}$$

with h_X and h_Y, functions of the invariant coordinates.

From the commutator (4.5), we deduce that:

$$[X, Y] + \frac{1}{h_Y}(X h_Y)Y - \frac{1}{h_X}(Y h_X)X = 0, \tag{4.6a}$$

and

$$X A_Y - Y A_X + [A_X, A_Y] + \frac{1}{h_Y}(X h_Y)A_Y - \frac{1}{h_X}(Y h_X)A_X = 0. \tag{4.6b}$$

The functions h_X and h_Y are determined by solving (4.6) with the requirement that (4.6b) corresponds to the reduced SDYM equations. Let us indicate that holonomic vector components to any reduced Lax pair can be found by introducing the above-mentioned factors : h_X and h_Y. For two- and three-dimensional vector fields, holonomic components can be determined by solving uniquely (4.6a), then (4.6b) will automatically coincide with the reduced SDYM equations.

We end this section by presenting two examples which illustrates the above method:

(1) $\{Y_3, P_3, P_4\}$:

The lifted vector fields have the form:

$$\tilde{Y}_3 = -\frac{1}{2}(x^1\partial_2 - x^2\partial_1 - x^3\partial_4 + x^4\partial_3) - i(\lambda\partial_\lambda - \bar{\lambda}\partial_{\bar{\lambda}}) = -\frac{1}{2}\partial_\varphi, \tag{4.7a}$$

with

$$\tilde{P}_3 = \partial_3, \tag{4.7b}$$

$$\tilde{P}_4 = \partial_4. \tag{4.7c}$$

Since the lift of Y_3 to CP^{3*} is nontrivial, we expect a new spectral parameter among the invariant variables. The orbit coordinates are:

$$\varphi = -\arctan\Big(\frac{x^2}{x^1}\Big) + \frac{i}{4}\ln\Big(\frac{\lambda}{\bar{\lambda}}\Big), \quad x^3, \quad x^4 \tag{4.8}$$

and the invariant coordinates can be chosen as:

$$r = \sqrt{(x^1)^2 + (x^2)^2}, \quad \Lambda = \lambda\bar{\lambda}$$

and

$$\eta = -\arctan\Big(\frac{x^2}{x^1}\Big) - \frac{i}{4}\ln\Big(\frac{\lambda}{\bar{\lambda}}\Big). \tag{4.9}$$

In terms of these variables on the stratum, the symmetric Yang-Mills fields have the form:

$$(A_1, A_2, A_3, A_4)^T = e^{-2\,\theta Y_3}(u_1, u_2, u_3, u_4)^T, \tag{4.10}$$

where $\theta = \dfrac{\eta+\varphi}{2}$, $u_\mu = u_\mu(r)$, $\mu = 1, ..., 4$ and $\Psi = \psi(r, \Lambda, \eta)$.

Inserting (4.8), (4.9), (4.10) and Ψ in the linear system (2.4), we find:

$$\Big[\partial_r - \frac{i}{r}\partial_\eta + u_1 + iu_2 - e^{i2\theta}\lambda(u_3 + iu_4)\Big]\psi = 0, \tag{4.11a}$$

$$\Big[e^{i2\theta}\lambda\,\Big(\partial_r + \frac{i}{r}\partial_\eta + u_1 - iu_2\Big) + u_3 - iu_4\Big]\psi = 0. \tag{4.11b}$$

The condition (2.4c): $\partial_{\bar{\lambda}}\Psi = 0$ or $\Big(\dfrac{\partial}{\partial\sqrt{\Lambda}} + \dfrac{i}{2\sqrt{\Lambda}}\dfrac{\partial}{\partial\eta}\Big)\,\psi = 0$, restricts us to the invariant $\zeta = \sqrt{\Lambda}e^{i2\eta}$. In terms of the new spectral parameter ζ, the equations (4.11) become the reduced Lax pair:

$$\nabla_X\Psi = \Big[\partial_r + \frac{2\zeta}{r}\partial_\zeta + u_1 + iu_2 - \zeta(u_3 + iu_4)\Big]\psi = 0, \tag{4.12a}$$

$$\nabla_Y\Psi = \Big[\zeta\Big(\partial_r - \frac{2\zeta}{r}\partial_\zeta\Big) + \zeta(u_1 - iu_2) + (u_3 - iu_4)\Big]\psi = 0. \tag{4.12b}$$

A Lax pair expressed in terms of holonomic vector fields is derived from (4.12) if $h_X \propto r$ and $h_Y \propto \frac{r}{\zeta}$. The SDYM equations reduced under the same subgroup arise as the compatibility of the linear system (4.12) [44]:

$$\begin{aligned} &\dot{u}_2 + \frac{u_2}{r} + [u_1, u_2] - [u_3, u_4] = 0, \\ &\dot{u}_3 - \frac{u_3}{r} + [u_1, u_3] + [u_2, u_4] = 0, \\ &\dot{u}_4 - \frac{u_4}{r} + [u_1, u_4] + [u_3, u_2] = 0, \end{aligned} \tag{4.13}$$

where a dotted variable indicates a differentiation with respect to its argument. With the change of variables: $\xi = \ln(r), w_2 = r\, u_2, w_3 = r^{-1}u_3, w_4 = r^{-1}u_4$, and the gauge condition $u_1 = 0$, the integrability of (4.12) leads to a modified form of the Nahm's equations [44].

To simplify computations in Example 2, the vector $P_{\bar{\lambda}} = \partial_{\bar{\lambda}}$ of the holomorphicity equation (2.4c) is included in the symmetry algebra of lifted vector fields.

(2) $\{X_3, Y_3\}$:

Using the holomorphicity condition or invariance along $\bar{\lambda}$: $\partial_{\bar{\lambda}}\Psi = 0$:

$$\tilde{X}_3 = -\frac{1}{2}(x^1\partial_2 - x^2\partial_1 + x^3\partial_4 - x^4\partial_3) = -\frac{1}{2}\partial_\chi, \tag{4.14a}$$

$$\tilde{Y}_3 = -\frac{1}{2}(x^1\partial_2 - x^2\partial_1 - x^3\partial_4 + x^4\partial_3) - i(\lambda\partial_\lambda - \bar{\lambda}\partial_{\bar{\lambda}}) = \frac{1}{2}\partial_\phi, \tag{4.14b}$$

where we have elected the orbit coordinates: χ, ϕ, and $\bar{\lambda}$ and the invariant variables: r, R, and ζ, which correspond to:

$$x^1 = r\cos\Big(\frac{\chi+\phi}{2}\Big)^4, \quad x^2 = -r\sin\Big(\frac{\chi+\phi}{2}\Big),$$

$$x^3 = R\cos\Big(\frac{\chi-\phi}{2}\Big), \quad x^4 = -R\sin\Big(\frac{\chi-\phi}{2}\Big)^4, \quad \lambda = e^{2i\phi}\zeta. \tag{4.15}$$

Here ζ stands for the new spectral parameter.

The invariant Yang-Mills field obeying to (3.1) is given by:

$$(A_1, A_2, A_3, A_4)^T = e^{-\chi X_3}e^{-\phi Y_3}(u_1, u_2, u_3, u_4)^T, \tag{4.16}$$

where $u_\mu = u_\mu(r, R)$ and $\Psi = \psi(r, R, \zeta)$.

Substitution of (4.15), (4.16), and Ψ into the Lax pair (2.4) implies that:

$$\nabla_X\Psi = \left[\partial_r - \zeta\partial_R + \left(\frac{\zeta}{r} + \frac{\zeta^2}{R}\right)\partial_\zeta + u_1 + iu_2 - \zeta(u_3 + iu_4)\right]\psi = 0, \tag{4.17a}$$

$$\nabla_Y\Psi = \left[\partial_R + \zeta\partial_r + \left(\frac{\zeta}{R} - \frac{\zeta^2}{r}\right)\partial_\zeta + u_3 - iu_4 + \zeta(u_1 - iu_2)\right]\psi = 0. \tag{4.17b}$$

The reduced SDYM equations are recovered through the compatibility of the system (4.17) and have the form:

$$\partial_r u_3 - \partial_R u_1 + [u_1, u_3] + [u_2, u_4] = 0, \tag{4.18a}$$

$$\partial_r u_4 + \partial_R u_2 - [u_2, u_3] + [u_1, u_4] = 0, \tag{4.18b}$$

$$\partial_R u_4 - \partial_r u_2 + \frac{u_4}{R} - \frac{u_2}{r} - [u_1, u_2] + [u_3, u_4] = 0. \tag{4.18c}$$

A Lax pair with holonomic vectors follows if we put $h_X \propto r$ and $h_Y \propto R$.

5 Reduced Lax and SDYM Equations

In this section, we present the resulting symmetry reductions with respect to the representatives of the classes of subalgebras of $e(4)$ (see Table 2 of Ref. [44]) giving rise to differential systems which are not equivalent to a zero curvature (without parameter) condition on the residual potentials, since the latter are then gauge equivalent to vanishing solutions (see for instance the subalgebra 4d), or to systems of uncoupled first order linear O.D.E.'s after a gauge choice (see for instance the representative 6b). One finds the list of representatives of these subalgebras in Table 1 of this article. For comparison, the Ref. [44] includes all the reduced SDYM equations under the representatives of $e(4)$. The labels attached to the representatives refer to the numbering adopted in Ref. [44], the generators or elements forming a basis are also specified within curly brackets. We then provide the orbit and invariant variables, as well as the invariant Yang-Mills fields determined according to section 3. The reduced Lax pairs and their compatibility condition, the reduced SDYM equations, follow. In order to simplify the computations, we have considered the vector $P_{\bar\lambda} = \partial_{\bar\lambda}$ of the holomorphicity equation (2.4c) as part of the algebra of lifted vector fields. This extension of the representative gives rise to the same reduced equations since the $P_{\bar\lambda}$-symmetry condition is equivalent to eq. (2.4c). We have parametrized the orbits of the translations generated under $P_{\bar\lambda}$ with the coordinate $\bar\lambda$. In the following, the reduced

T A B L E 1
Representatives of Conjugacy Classes of Subalgebras of $e(4)$
Leading to Nonlinear Reduced SDYM Equations†

Representative(s) # in ref. [44]	Basis of Subalgebra	Condition
1a	P_4	
1b	P_3, P_4	
1c	P_1, P_2, P_3	
2a,3a,4a,5a	$\alpha X_3 + \beta Y_3$	$\alpha, \beta \in R$
2b,3b,4c,5b	$\alpha X_3 + \beta Y_3, P_3, P_4$	$\alpha, \beta \in R$
4b	$X_3 + Y_3, P_3$	
6a	X_3, Y_3	
8a	X_1, X_2, X_3	
9a	Y_1, Y_2, Y_3	
13a	$X_3 + Y_3 + cP_4$	$c \in R$
13b	$X_3 + Y_3 + cP_4, P_3$	$c \in R$
13c	$X_3 + Y_3 + cP_4, P_1, P_2$	$c \in R$

†The reduced SDYM equations are not equivalent to a zero curvature condition without parameter, or to a system of uncoupled linear first order O.D.E.'s.

equations are first written without any special choice of gauge. In some cases, a relation to already known integrable systems is indicated.

1. 1a $\{P_4\}$

Orbit coordinates: $x^4, \bar{\lambda}$.
Invariant coordinates: x^1, x^2, x^3, λ.
Invariant A_μ and Ψ:

$$A_\mu = u_\mu(x^1, x^2, x^3)^4, \quad \Psi = \psi(x^1, x^2, x^3, \lambda). \tag{5.1}$$

Reduced Lax pair:

$$\begin{aligned} &[\partial_1 + i\partial_2 + u_1 + iu_2 - \lambda(\partial_3 + u_3 + iu_4)]\psi = 0, \\ &[\partial_3 + u_3 - iu_4 + \lambda(\partial_1 - i\partial_2 + u_1 - iu_2)]\psi = 0. \end{aligned} \tag{5.2}$$

Reduced SDYM equations:

$$\begin{aligned} &\partial_1 u_2 - \partial_1 u_2 - \partial_3 u_4 + [u_1, u_2] - [u_3, u_4] = 0, \\ &\partial_2 u_3 - \partial_3 u_2 - \partial_1 u_4 + [u_2, u_3] - [u_1, u_4] = 0, \\ &\partial_1 u_3 - \partial_3 u_1 + \partial_2 u_4 + [u_1, u_3] + [u_2, u_4] = 0, \end{aligned} \tag{5.3}$$

which correspond to the Bogomolny equations [48, 49] with $u_4 = \phi$.

2. 1b $\{P_3, P_4\}$

Orbit coordinates: $x^3, x^4, \bar{\lambda}$.
Invariant coordinates: x^1, x^2, λ.
Invariant A_μ and Ψ:

$$A_\mu = u_\mu(x^1, x^2)^4, \quad \Psi = \psi(x^1, x^2, \lambda). \tag{5.4}$$

Reduced Lax pair:

$$\begin{aligned} &[\partial_1 + i\partial_2 - \lambda(u_3 + iu_4) + u_1 + iu_2]\psi = 0, \\ &[\lambda(\partial_1 - i\partial_2 + u_1 - iu_2) + u_3 - iu_4]\psi = 0. \end{aligned} \tag{5.5}$$

Reduced SDYM equations:

$$\begin{aligned} &\partial_1 u_2 - \partial_1 u_2 + [u_1, u_2] - [u_3, u_4] = 0, \\ &\partial_2 u_3 - \partial_1 u_4 + [u_2, u_3] - [u_1, u_4] = 0, \\ &\partial_1 u_3 + \partial_2 u_4 + [u_1, u_3] + [u_2, u_4] = 0. \end{aligned} \tag{5.6}$$

A number of algebraic reductions have been performed with the help of gauge choices starting from eq.(5.6). For instance, one can find a reduction to the Toda lattice equations, to the chiral field equations (if null variables are used), as well as the elliptic sine-Gordon equation [50].

3. 1c $\{P_1, P_2, P_3\}$

Orbit coordinates: $x^1, x^2, x^3, \bar{\lambda}$.

Invariant coordinates: x^4, λ.
Invariant A_μ and Ψ:

$$A_\mu = u_\mu (x^4)^4, \quad \Psi = \psi(x^4, \lambda). \tag{5.7}$$

Reduced Lax pair:

$$\begin{aligned} &[\lambda(i\partial_4 + u_3 + iu_4) - u_1 - iu_2]\psi = 0, \\ &[i\partial_4 - \lambda(u_1 - iu_2) - u_3 + iu_4]\psi = 0. \end{aligned} \tag{5.8}$$

Reduced SDYM equations:

$$\begin{aligned} &\partial_4 u_1 + [u_2, u_3] - [u_1, u_4] = 0, \\ &\partial_4 u_2 - [u_1, u_3] - [u_2, u_4] = 0, \\ &\partial_4 u_3 + [u_1, u_2] - [u_3, u_4] = 0. \end{aligned} \tag{5.9}$$

The Nahm equations [51–53] are recovered with the gauge choice $u_4 = 0$ and $u_a \to -u_a$.

4. 2a,3a,4a,5a $\{\alpha X_3 + \beta Y_3 | \alpha, \beta \in R\}$

Orbit coordinates: $\xi = -(\alpha+\beta)\arctan\left(\frac{x^2}{x^1}\right) - (\alpha-\beta)\arctan\left(\frac{x^4}{x^3}\right), \bar{\lambda}$.
Invariant coordinates:

$$\chi = (\alpha-\beta)\arctan\left(\frac{x^2}{x^1}\right) - (\alpha+\beta)\arctan\left(\frac{x^4}{x^3}\right), \quad r = \sqrt{(x^1)^2 + (x^2)^2},$$

$$R = \sqrt{(x^3)^2 + (x^4)^2}, \quad \zeta = e^{(i\gamma(-(\alpha+\beta)\arctan(x^2/x^1) - (\alpha-\beta)\arctan(x^4/x^3)))}\lambda = e^{i\gamma\xi}\lambda,$$

where $\gamma = \frac{\beta}{\alpha^2+\beta^2}$.
Invariant A_μ and Ψ:

$$\begin{aligned} &A_1 = u_1\cos\theta + u_2\sin\theta, \quad A_2 = -u_1\sin\theta + u_2\cos\theta, \\ &A_3 = u_3\cos\phi + u_4\sin\phi, \quad A_4 = -u_3\sin\phi + u_4\cos\phi, \end{aligned} \tag{5.10}$$

with $\phi = \frac{(\alpha+\beta)\chi + (\alpha-\beta)\xi}{2(\alpha^2+\beta^2)}$, $\theta = \frac{-(\alpha-\beta)\chi + (\alpha+\beta)\xi}{2(\alpha^2+\beta^2)}$, $u_\mu = u_\mu(r, R, \chi)$ and $\Psi = \psi(r, R, \chi, \zeta)$.

Reduced Lax pair:

$$\left[\partial_r - \zeta e^{-i\Gamma\chi}\partial_R + i\left(\frac{\alpha-\beta}{r} + \frac{(\alpha+\beta)}{R}\zeta e^{-i\Gamma\chi}\right)\partial_\chi + \left(\frac{\gamma(\alpha+\beta)}{r} - \frac{\gamma(\alpha-\beta)}{R}\zeta e^{-i\Gamma\chi}\right)\times \right.$$
$$\left. \zeta\partial_\zeta + u_1 + iu_2 - \zeta e^{-i\Gamma\chi}(u_3 + iu_4)\right]\psi = 0, \tag{5.11a}$$

$$\left[\partial_R + \zeta e^{-i\Gamma\chi}\partial_r + i\left(\frac{\alpha+\beta}{R} - \frac{(\alpha-\beta)}{r}\zeta e^{-i\Gamma\chi}\right)\partial_\chi - \left(\frac{\gamma(\alpha-\beta)}{R} + \frac{\gamma(\alpha+\beta)}{r}\zeta e^{-i\Gamma\chi}\right)\times \right.$$
$$\left. \zeta\partial_\zeta + u_3 - iu_4 + \zeta e^{-i\Gamma\chi}(u_1 - iu_2)\right]\psi = 0, \tag{5.11b}$$

where $\gamma = \frac{\beta}{\alpha^2+\beta^2}$ and $\Gamma = \frac{\alpha}{\alpha^2+\beta^2}$.

Let us note that holonomic vector parts can be obtained if (5.11a) and (5.11b) are respectively multiplied by r and R.

Reduced SDYM equations:

$$\begin{gathered}\partial_R u_4 - \partial_r u_2 + \frac{u_4}{R} - \frac{u_2}{r} + \frac{(\alpha+\beta)}{R}\partial_\chi u_3 + \frac{(\alpha-\beta)}{r}\partial_\chi u_1 + [u_2,u_1] + [u_3,u_4] = 0,\\ \partial_r u_3 - \partial_R u_1 + \frac{(\alpha-\beta)}{r}\partial_\chi u_4 + \frac{(\alpha+\beta)}{R}\partial_\chi u_2 + [u_1,u_3] + [u_2,u_4] = 0, \\ \partial_r u_4 + \partial_R u_2 - \frac{(\alpha-\beta)}{r}\partial_\chi u_3 + \frac{(\alpha+\beta)}{R}\partial_\chi u_1 + [u_2,u_3] - [u_1,u_4] = 0.\end{gathered} \tag{5.12}$$

5. 2b,3b,4c,5b $\{\alpha X_3 + \beta Y_3, P_3, P_4 | \alpha, \beta \in R\}$

Orbit coordinates: $\theta = -\arctan\left(\frac{x^2}{x^1}\right)$, $x^3, x^4, \bar{\lambda}$.

Invariant coordinates: $r, \zeta = e^{i\gamma\theta}\lambda$, where $\gamma = \dfrac{2\beta}{\alpha+\beta}$.

Invariant A_μ and Ψ:

$$\begin{gathered}A_1 = u_1\cos\theta + u_2\sin\theta, \quad A_2 = -u_1\sin\theta + u_2\cos\theta,\\ A_3 = u_3\cos\Big(\frac{\alpha-\beta}{\alpha+\beta}\theta\Big) + u_4\sin\Big(\frac{\alpha-\beta}{\alpha+\beta}\theta\Big), \quad A_4 = -u_3\sin\Big(\frac{\alpha-\beta}{\alpha+\beta}\theta\Big) + u_4\cos\Big(\frac{\alpha-\beta}{\alpha+\beta}\theta\Big),\end{gathered} \tag{5.13}$$

where $u_\mu = u_\mu(r)$ and $\Psi = \psi(r,\zeta)$.

Reduced Lax pair:

$$\begin{gathered}\Big[\partial_r + \frac{\gamma}{r}\zeta\partial_\zeta + u_1 + iu_2 - \zeta(u_3 + iu_4)\Big]\psi = 0,\\ \Big[\zeta\partial_r - \frac{\gamma}{r}\zeta^2\partial_\zeta + \zeta(u_1 - iu_2) + u_3 - iu_4\Big]\psi = 0.\end{gathered} \tag{5.14}$$

Holonomic vector components can be obtained if the two above equations are multiplied by r.

Reduced SDYM equations:

$$\begin{gathered}\frac{du_2}{dr} + \frac{u_2}{r} + [u_1,u_2] - [u_3,u_4] = 0,\\ \frac{du_3}{dr} + \frac{1-\gamma}{r}u_3 + [u_1,u_3] + [u_2,u_4] = 0,\\ \frac{du_4}{dr} + \frac{1-\gamma}{r}u_4 + [u_1,u_4] + [u_3,u_2] = 0.\end{gathered} \tag{5.15}$$

With a suitable change of variables and gauge choice, the above SDYM and corresponding Lax equations can be algebraically reduced respectively to the equations of the Toda lattice with damping and its linear system [33].

6. 4b $\{X_3 + Y_3, P_3\}$

Orbit coordinates: $\theta = -\arctan\Big(\frac{x^2}{x^1}\Big), x^3, \bar{\lambda}$.

Invariant coordinates: $r = \sqrt{(x^1)^2 + (x^2)^2},\ x^4,\ \ \zeta = e^{i\theta}\lambda$.

Invariant A_μ and Ψ:

$$A_1 = u_1 \cos\theta + u_2 \sin\theta,\ A_2 = -u_1 \sin\theta + u_2 \cos\theta,\ A_3 = u_3,\ A_4 = u_4 \tag{5.16}$$

where $u_\mu = u_\mu(r, x^4)$ and $\Psi = \psi(r, x^4, \zeta)$.

Reduced Lax pair:

$$\Big[\partial_r + \frac{1}{r}\zeta\partial_\zeta - i\zeta\partial_4 + u_1 + iu_2 - \zeta(u_3 + iu_4)\Big]\psi = 0,$$
$$\Big[\zeta\partial_r - \frac{1}{r}\zeta^2\partial_\zeta - i\partial_4 + \zeta(u_1 - iu_2) + u_3 - iu_4\Big]\psi = 0. \tag{5.17}$$

One can find holonomic vector components if the two above equations are respectively multiplied by r and $\frac{r}{\zeta}$.

Reduced SDYM equations:

$$\begin{aligned}
&\partial_r u_2 + \partial_4 u_3 + \frac{u_2}{r} + [u_1, u_2] - [u_3, u_4] = 0,\\
&\partial_r u_3 - \partial_4 u_2 + [u_1, u_3] + [u_2, u_4] = 0,\\
&\partial_r u_4 - \partial_4 u_1 + [u_1, u_4] + [u_3, u_2] = 0.
\end{aligned} \tag{5.18}$$

7. 6a $\{X_3, Y_3\}$

Orbit coordinates:

$$\xi = -\arctan\Big(\frac{x^2}{x^1}\Big) - \arctan\Big(\frac{x^4}{x^3}\Big), \quad \chi = -\arctan\Big(\frac{x^2}{x^1}\Big) + \arctan\Big(\frac{x^4}{x^3}\Big),\ \bar\lambda.$$

Invariant coordinates: $r = \sqrt{(x^1)^2 + (x^2)^2}$, $R = \sqrt{(x^3)^2 + (x^4)^2}$, $\zeta = e^{i\chi}\lambda$.

Invariant A_μ and Ψ.:

$$A_1 = u_1 \cos\Big(\frac{\xi+\chi}{2}\Big) + u_2 \sin\Big(\frac{\xi+\chi}{2}\Big), \quad A_2 = -u_1 \sin\Big(\frac{\xi+\chi}{2}\Big) + u_2 \cos\Big(\frac{\xi+\chi}{2}\Big),$$
$$A_3 = u_3 \cos\Big(\frac{\xi-\chi}{2}\Big) + u_4 \sin\Big(\frac{\xi-\chi}{2}\Big), \quad A_4 = -u_3 \sin\Big(\frac{\xi-\chi}{2}\Big) + u_4 \cos\Big(\frac{\xi-\chi}{2}\Big), \tag{5.19}$$

where $u_\mu = u_\mu(r, R)$ and $\Psi = \psi(r, R, \zeta)$.

The reduced Lax pair and SDYM equations are deduced from the same equations obtained in case 4 with the values $\alpha = 1$ and $\beta = 0$ by setting $\partial_\chi\psi = 0$ and $\partial_\chi u_\mu = 0$.

8. 8a $\{X_1, X_2, X_3\}$

Orbit coordinates: ϕ_1, ϕ_2, ϕ_3 in $x = e^{(\phi_1-\phi_2)X_2} e^{\phi_3 X_1} e^{(\phi_1+\phi_2)X_2}[0, 0, 0, R]^T$, and $\bar\lambda$.

Invariant coordinates: $R = \sqrt{x^\mu x_\mu}, \lambda$.

Invariant A_μ and Ψ:

$$[A_1, A_2, A_3, A_4]^T = e^{(\phi_1-\phi_2)X_2} e^{\phi_3 X_1} e^{(\phi_1+\phi_2)X_2}[u_1, u_2, u_3, u_4]^T, \tag{5.20a}$$

where $u_\mu = u_\mu(R)$. For the purpose of the calculations, we can express it in terms of Cartesian coordinates. We then have:

$$A_\mu = 2(\bar\eta^a_{\mu\nu} x^\nu v_a + \delta_{\mu\nu} x^\nu v_4), \tag{5.20b}$$

where $v_\mu = v_\mu(\mathcal{R})$, with $\mathcal{R} = R^2$ and $v_a = -\frac{1}{2R}u_a\,(a = 1,2,3), v_4 = \frac{1}{2R}u_4$ and $\Psi = \psi(\mathcal{R},\lambda)$.

Reduced Lax pair:

$$\mathcal{R}^2[\partial_{\mathcal{R}} + v_4 - iv_3 - \lambda(iv_1 + v_2)]\psi = 0,$$
$$\mathcal{R}^2[\lambda\partial_{\mathcal{R}} - iv_1 + v_2 + \lambda(iv_3 + v_4)]\psi = 0. \tag{5.21}$$

Reduced SDYM equations:

$$\begin{aligned}\partial_{\mathcal{R}}v_1 + \frac{2}{\mathcal{R}}v_1 + [v_2, v_3] + [v_1, v_4] &= 0,\\ \partial_{\mathcal{R}}v_2 + \frac{2}{\mathcal{R}}v_2 + [v_1, v_3] - [v_2, v_4] &= 0,\\ \partial_{\mathcal{R}}v_3 + \frac{2}{\mathcal{R}}v_3 - [v_1, v_2] - [v_3, v_4] &= 0.\end{aligned} \tag{5.22}$$

Let us add that the Nahm's equations can be retrieved by putting $v_4 = 0$ and by carrying out the following change of variables: $\varphi = -\frac{1}{2\mathcal{R}}$ and $w_a = -2\mathcal{R}^2 v_a$.

9. 9a $\{Y_1, Y_2, Y_3\}$

Orbit coordinates: ϕ_1, ϕ_2, ϕ_3 in $x = e^{(\phi_1-\phi_2)Y_2}e^{\phi_3 Y_1}e^{(\phi_1+\phi_2)Y_2}[0,0,0,R]^T$, and $\bar{\lambda}$.

Invariant coordinates: $R = \sqrt{x^\mu x_\mu}$, $\zeta = \frac{z^1 - \lambda\bar{z}^2}{z^2 + \lambda\bar{z}^1}$, where $z^1 := x^1 + ix^2$ and $z^2 := x^3 - ix^4$.

Invariant A_μ and Ψ:

$$[A_1, A_2, A_3, A_4]^T = e^{(\phi_1-\phi_2)Y_2}e^{\phi_3 Y_1}e^{(\phi_1+\phi_2)Y_2}[u_1, u_2, u_3, u_4]^T, \tag{5.23a}$$

where $u_\mu = u_\mu(R)$. In order to facilitate calculations, it can be rewritten in terms of Cartesian coordinates:

$$A_\mu = 2(\eta^a_{\mu\nu}x^\nu v_a + \delta_{\mu\nu}x^\nu v_4), \tag{5.23b}$$

where $v_\mu = v_\mu(\mathcal{R})$, with $\mathcal{R} = R^2$ and $v_\mu = -\frac{1}{2R}u_\mu$ and $\Psi = \psi(R,\zeta)$.

Reduced Lax pair:

$$[\partial_{\mathcal{R}} + iv_3 + v_4 + \zeta(v_2 + iv_1)]\psi = 0,$$
$$[\zeta\partial_{\mathcal{R}} + iv_1 - v_2 + \zeta(v_4 - iv_3)]\psi = 0. \tag{5.24}$$

Reduced SDYM equations:

$$\begin{aligned}[\partial_{\mathcal{R}}v_1 + [v_2, v_3] - [v_1, v_4]]\psi &= 0,\\ [\partial_{\mathcal{R}}v_2 + [v_3, v_1] - [v_2, v_4]]\psi &= 0,\\ [\partial_{\mathcal{R}}v_3 + [v_1, v_2] - [v_3, v_4]]\psi &= 0.\end{aligned} \tag{5.25}$$

The Nahm's equations are derived if we require $v_4 = 0$ and change v_a to $-v_a$. Contrary to the previous cases, we would like to point out that even if the lift of the elements of the symmetry algebra is nontrivial, the reduced Lax pair does not involve vector components in the direction of the new spectral parameter.

10. 13a $\{X_3 + Y_3 + cP_4\}$

Orbit coordinates: $\xi = -\arctan\left(\frac{x^2}{x^1}\right) - cx^4, \bar{\lambda}$.

Invariant coordinates: $r = \sqrt{(x^1)^2 + (x^2)^2}, x^3, \quad \chi = -c\arctan\left(\frac{x^2}{x^1}\right) + x^4, \quad \zeta = e^{i\gamma\xi}\lambda$, where $\gamma = \frac{1}{1+c^2}$.

Invariant A_μ and Ψ:

$$A_1 = u_1\cos(\gamma\xi) + u_2\sin(\gamma\xi),\ A_2 = -u_1\sin(\gamma\xi) + u_2\cos(\gamma\xi),\ A_3 = u_3,\ A_4 = u_4, \tag{5.26}$$

where $u_\mu = u_\mu(r, x^3, \chi)$ and $\Psi = \psi(r, x^3, \chi, \zeta)$.

Reduced Lax pair:

$$\begin{gathered}\Big[\partial_r - i\Big(e^{i\gamma c\chi}\zeta + \frac{c}{r}\Big)\partial_\chi - \zeta e^{i\gamma c\chi}\partial_3 + \Big(\frac{\gamma}{r} - \gamma c e^{i\gamma c\chi}\zeta\Big)\zeta\partial_\zeta + e^{i\gamma c\chi}((u_1 + iu_2) - \\ \zeta(u_3 + iu_4))\Big]\psi = 0, \\ \Big[e^{i\gamma c\chi}\zeta\partial_r + \partial_3 + i\Big(e^{i\gamma c\chi}\frac{c}{r}\zeta - 1\Big)\partial_\chi - \Big(\frac{\gamma}{r}\zeta e^{i\gamma c\chi} + \gamma c\Big)\zeta\partial_\zeta + \zeta(u_1 - iu_2) + u_3 - iu_4\Big]\psi = 0.\end{gathered} \tag{5.27}$$

The above linear system is composed of holonomic vectors if the first equation is multiplied by r.

Reduced SDYM equations:

$$\begin{aligned}&\sin(\gamma c\chi)\Big(\partial_r u_1 + \frac{\gamma}{r}u_1 - \frac{c}{r}\partial_\chi u_2\Big) + \cos(\gamma c\chi)\Big(\partial_r u_2 + \frac{\gamma}{r}u_2 + \frac{c}{r}\partial_\chi u_1\Big) - \\ &\qquad \partial_3 u_4 + \partial_\chi u_3 + [u_1, u_2] - [u_3, u_4] = 0, \\ &\cos(\gamma c\chi)\Big(\partial_r u_3 - \frac{c}{r}\partial_\chi u_4\Big) - \sin(\gamma c\chi)\Big(\partial_r u_4 + \frac{c}{r}\partial_\chi u_3\Big) - \\ &\qquad \partial_3 u_1 - c\gamma u_1 - \partial_\chi u_2 + [u_1, u_3] + [u_2, u_4] = 0, \\ &\sin(\gamma c\chi)\Big(\partial_r u_3 - \frac{c}{r}\partial_\chi u_4\Big) + \cos(\gamma c\chi)\Big(\partial_r u_4 + \frac{c}{r}\partial_\chi u_3\Big) + \\ &\qquad \partial_3 u_2 + c\gamma u_2 - \partial_\chi u_1 - [u_2, u_3] + [u_1, u_4] = 0.\end{aligned} \tag{5.28}$$

11. 13b $\{X_3 + Y_3 + cP_4, P_3\}$

The reduced Lax pair and SDYM equations are obtained by ignoring any dependence with respect to the orbit variable x^3 in the equations (5.27) and (5.28).

12. 13c $\{X_3 + Y_3 + cP_4, P_1, P_2\}$

Orbit coordinates: $x^1, x^2, \theta = \frac{x^4}{c}, \bar{\lambda}$.

Invariant coordinates: $x^3, \zeta = e^{-i(x^4/c)}\lambda$.

Invariant A_μ and Ψ:

$$A_1 = u_1,\ A_2 = u_2,\ A_3 = u_3,\ A_4 = u_4, \tag{5.29}$$

where $u_\mu = u_\mu(x^3)$ and $\Psi = \psi(x^3, \zeta)$.

Reduced Lax pair:

$$\left[\zeta\partial_3 + \frac{\zeta^2}{c}\partial_\zeta + \zeta(u_3 + iu_4) - u_1 - iu_2\right]\psi = 0,$$
$$\left[\partial_3 - \frac{\zeta}{c}\partial_\zeta + \zeta(u_1 - iu_2) + u_3 - iu_4\right]\psi = 0. \qquad (5.30)$$

We have a set of holonomic vector fields for this linear system if the factor $\frac{1}{\zeta}$ is added to the first equation.

Reduced SDYM equations:

$$\begin{gathered}\partial_3 u_1 + \frac{u_1}{c} - [u_1, u_3] - [u_2, u_4] = 0,\\ \partial_3 u_2 + \frac{u_2}{c} - [u_2, u_3] + [u_1, u_4] = 0,\\ \partial_3 u_4 - [u_1, u_2] + [u_3, u_4] = 0.\end{gathered} \qquad (5.31)$$

6 Conclusion

In this paper, we have applied the method of reduction by symmetry to the Lax pair, or linear system, of the SDYM equations on four-dimensional Euclidean space. Two main aspects to be considered were, first, the extension of the Lax pair to the product of the Euclidean space and the space of the spectral parameter (CP^1), and second, the lift of the group action of $SO(5,1)$ to $R^4 \times CP^1 \subset CP^{3*}$ preserving the complex structures induced on E^4 by the linear system (2.4). Using a classification of the subalgebras of $e(4)$ under conjugacy classes with respect to the adjoint action of $E_o(4)$, we have reduced the Lax pair for the SDYM equations under each class representative which produces a nontrivial residual differential system. A list of these representatives can be read in Table 1 and the reduced Lax pairs are given in section 5. The compatibility of the reduced Lax pairs agrees exactly with the similarly reduced SDYM equations. For many reduced linear systems, a vector component along the (new) spectral parameter arose, typically when a nontrivial lift of the group action was involved.

As possible developments of this work, further reductions of the Lax pairs and SDYM equations can be effected for the representatives of conjugacy classes of subgroups of $SO(5,1)$ as well as for Yang-Mills fields (A_μ) invariant up to gauge transformations. One can also carry out reductions of the same set of equations on $R^{(2,2)}$ with respect to subgroups of the corresponding conformal group $SO(3,3)$, and equally for higher-dimensional and self-dual spaces versions of these equations under subgroups of their space transformation groups. Finally, it could be interesting to apply the method of symmetry reduction to the (universal) hierarchy of SDYM equations and to supersymmetric generalizations of the above systems.

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