

Multiparameter Deformations of the Algebra gl_n in Terms of Anyonic Oscillators

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Abstract

Generators of multiparameter deformations $U_{q;s_1,s_2,\dots,s_{n-1}}(gl_n)$ of the universal enveloping algebra $U(gl_n)$ are realized bilinearly by means of an appropriately generalized form of anyonic oscillators (AOs). This modification takes into account the parameters s_1, \dots, s_{n-1} and yields usual AOs when all the s_i are set equal to unity.

1. Various aspects of quantum groups and quantum (or q -deformed) algebras [1, 2] remain to be a subject of intensive study. Recently, it was shown by Lerda and Sciuto [3] that the q -algebra $U_q(su_2)$ admits a realization in terms of two modes of so-called anyonic oscillators—certain nonlocal objects defined on a two-dimensional square lattice. Shortly after, this result was extended in [4]–[5] to the case of higher rank algebras $U_q(sl_n)$ and, moreover, to the q -analogs of all semisimple Lie algebras from the classical series A_r , B_r , C_r , and D_r .

During last few years, *multiparameter* (and two-parameter, in particular) deformations of GL_n groups and of the algebras $U(gl_n)$, $U(sl_n)$ were developed [6]–[11]. Accordingly, it is of interest to explore the possibility of constructing an analogs of anyonic realization for those multiparametric-deformed algebras. As a step in this direction, Matheus-Valle and R-Monteiro [12] have presented a kind of anyonic construction for the two parametric-deformed algebra $U_{p,q}(sl_2)$.

The subject of our consideration (just in the context of 'anyonic' realizations) will be a class of multiparameter deformations $U_{q;s_1,s_2,\dots,s_{n-1}}(gl_n)$ of the universal enveloping algebras $U(gl_n)$ which are generated by the elements $\mathbf{1}$, I_{jj+1} , I_{j+1j} , $j = 1, 2, \dots, n-1$, and I_{ii} , $i = 1, 2, \dots, n$, and are defined by the relations

$$\begin{aligned} [I_{ii}, I_{jj}] &= 0, \\ [I_{ii}, I_{jj+1}] &= \delta_{ij} I_{ij+1} - \delta_{ij+1} I_{ji}, \\ [I_{ii}, I_{j+1j}] &= \delta_{ij+1} I_{ij} - \delta_{ij} I_{j+1i}, \\ [I_{i+1i}, I_{jj+1}] &= [I_{i+1i}, I_{j+1j}] = 0 \quad \text{for } |i-j| \geq 2, \\ [I_{i+1i}, I_{j+1j}] &= \delta_{ij} \frac{(s_i q)^{I_{i+1,i+1}} (s_i^{-1} q)^{-I_{ii}} - (s_i q)^{I_{ii}} (s_i^{-1} q)^{-I_{i+1,i+1}}}{s_i^{-1} q - (s_i q)^{-1}}, \end{aligned} \tag{1}$$

together with the trilinear (or multiparameter generalization of q -Serre) relations

$$\begin{aligned}
 & [[I_{i,i+1}, I_{i+1,i+2}]_{(s_{i+1}q)}, I_{i,i+1}]_{(s_{i+1}^{-1}q)} = 0, \\
 & [[I_{i+1,i+2}, I_{i,i+1}]_{(s_{i+1}^{-1}q)}, I_{i+1,i+2}]_{(s_{i+1}q)} = 0, \\
 & [[I_{i+1,i}, I_{i+2,i+1}]_{(s_i^{-1}q)}, I_{i+1,i}]_{(s_iq)} = 0, \\
 & [[I_{i+2,i+1}, I_{i+1,i}]_{(s_iq)}, I_{i+2,i+1}]_{(s_i^{-1}q)} = 0.
 \end{aligned} \tag{2}$$

Here the notation $[X, Y]_r \equiv XY - r YX$ for a deformed commutator is adopted. With comultiplication rules

$$\begin{aligned}
 \Delta(I_{kk}) &= I_{kk} \otimes \mathbf{1} + \mathbf{1} \otimes I_{kk}, \\
 \Delta(I_{k,k+1}) &= I_{k,k+1} \otimes (s_k^{-1}q)^{-I_{kk}} q^{(I_{kk}+I_{k+1,k+1})/2} + \\
 &\quad (s_k q)^{I_{kk}} q^{-(I_{kk}+I_{k+1,k+1})/2} \otimes I_{k,k+1}, \\
 \Delta(I_{k+1,k}) &= I_{k+1,k} \otimes (s_k q)^{I_{k+1,k+1}} q^{-(I_{kk}+I_{k+1,k+1})/2} + \\
 &\quad (s_k^{-1}q)^{-I_{k+1,k+1}} q^{(I_{kk}+I_{k+1,k+1})/2} \otimes I_{k+1,k},
 \end{aligned} \tag{3}$$

and the counit structure

$$\epsilon(\mathbf{1}) = 1, \quad \epsilon(I_{kk}) = \epsilon(I_{k,k+1}) = \epsilon(I_{k+1,k}) = 0, \tag{4}$$

the considered algebra becomes a bialgebra. In the restricted case characterized by

$$s_1 = s_2 = \dots = s_{n-1} = s, \tag{5}$$

the algebra $U_{q;s_1,s_2,\dots,s_{n-1}}(gl_n)$ reduces to the two-parameter deformation $U_{q,s}(gl_n)$ given by Takeuchi [9]. If in addition the restriction $s = 1$ is imposed, the standard q -deformation of Drinfeld–Jimbo [1, 2] is recovered. Conversely, the algebra under consideration with defining relations (1)–(4) can be generated from the standard $U_q(gl_n)$ by applying the procedure described in [10] (see also [11]).

Below, we will present a realization of these multiparametric-deformed algebras $U_{q;s_1,s_2,\dots,s_{n-1}}(gl_n)$, as defined in relations (1)–(4), by means of a definite set of n modified anyonic oscillators (or ‘quasi-anyons’, i.e., an appropriate generalization of anyons used in [3]–[5]).

2. Let us begin with some preliminaries concerning ($d = 2$) a lattice angle function, disorder operator, and anyonic oscillators (see [3]–[5]). Let Ω be a two-dimensional square lattice with the spacing $a = 1$. On this lattice, we consider a set of N species (sorts) of fermions $c_i(\mathbf{x})$, $i = 1, \dots, N$, $\mathbf{x} \in \Omega$, which satisfy the following standard anticommutation relations:

$$\{c_i(\mathbf{x}), c_j(\mathbf{y})\} = \{c_i^\dagger(\mathbf{x}), c_j^\dagger(\mathbf{y})\} = 0, \tag{6}$$

$$\{c_i(\mathbf{x}), c_j^\dagger(\mathbf{y})\} = \delta_{ij} \delta(\mathbf{x}, \mathbf{y}). \tag{7}$$

Here $\delta(\mathbf{x}, \mathbf{y})$ is nothing but the conventional lattice δ -function: $\delta(\mathbf{x}, \mathbf{y}) = 1$ if $\mathbf{x} = \mathbf{y}$ and vanishes if $\mathbf{x} \neq \mathbf{y}$.

We use the same definition as in [3]–[5] of the lattice angle functions $\Theta_{\gamma_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})$ and $\Theta_{\delta_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})$ that correspond to the two opposite types of cuts (γ -type and δ -type), and the same definition of ordering ($\mathbf{x} > \mathbf{y}$, $\mathbf{x} < \mathbf{y}$). The corresponding two types of *disorder operators* $K_i(\mathbf{x}_\gamma)$, and $K_i(\mathbf{x}_\delta)$, $i = 1, \dots, N$, are introduced as follows:

$$\begin{aligned} K_j(\mathbf{x}_\gamma) &= \exp \left(i\nu \sum_{\mathbf{y} \neq \mathbf{x}} \Theta_{\gamma_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) c_j^\dagger(\mathbf{y}) c_j(\mathbf{y}) \right), \\ K_j(\mathbf{x}_\delta) &= \exp \left(i\nu \sum_{\mathbf{y} \neq \mathbf{x}} \Theta_{\delta_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) c_j^\dagger(\mathbf{y}) c_j(\mathbf{y}) \right). \end{aligned} \quad (8)$$

The number ν that appears here is usually called the *statistics parameter*.

The anyonic oscillators (AOs) $a_i(\mathbf{x}_\gamma)$ and $a_i(\mathbf{x}_\delta)$, $i = 1, \dots, N$, are defined [3] as

$$a_i(\mathbf{x}_\gamma) = K_i(\mathbf{x}_\gamma) c_i(\mathbf{x}), \quad a_i(\mathbf{x}_\delta) = K_i(\mathbf{x}_\delta) c_i(\mathbf{x}) \quad (9)$$

(no summation over i). One can show that these AOs satisfy the following relations of permutation. For $i \neq j$ and arbitrary $\mathbf{x}, \mathbf{y} \in \Omega$,

$$\{a_i(\mathbf{x}_\gamma), a_j(\mathbf{y}_\gamma)\} = \{a_i(\mathbf{x}_\gamma), a_j^\dagger(\mathbf{y}_\gamma)\} = 0. \quad (10)$$

Let $q = \exp(i\pi\nu)$. For $i = j$ and for two distinct sites (i.e., $\mathbf{x} \neq \mathbf{y}$) on the lattice Ω , one has

$$a_i(\mathbf{x}_\gamma) a_i(\mathbf{y}_\gamma) + q^{-\text{sgn}(\mathbf{x}-\mathbf{y})} a_i(\mathbf{y}_\gamma) a_i(\mathbf{x}_\gamma) = 0, \quad (11)$$

$$a_i(\mathbf{x}_\gamma) a_i^\dagger(\mathbf{y}_\gamma) + q^{\text{sgn}(\mathbf{x}-\mathbf{y})} a_i^\dagger(\mathbf{y}_\gamma) a_i(\mathbf{x}_\gamma) = 0, \quad (12)$$

whereas on the same site

$$(a_i(\mathbf{x}_\gamma))^2 = 0, \quad \{a_i(\mathbf{x}_\gamma), a_i^\dagger(\mathbf{x}_\gamma)\} = 1. \quad (13)$$

The analogs of relations (10)–(13) for anyonic oscillators of the opposite type δ are obtained from (10)–(13) by replacing $\gamma \rightarrow \delta$ and $q \rightarrow q^{-1}$.

Note that it is the pair of relations (11), (12) (their analogs for the δ -type of cut, and Hermitian conjugates of all them) which the statistics parameter ν does enter. In comparison with ordinary fermions, the basic feature of anyons is their nonlocality (the attributed cut) and their braiding property specific of $d = 2$ and given by eq. (11), (12). These latter imply that anyons of the *same sort*, even allocated at different sites of the lattice, nevertheless 'feel' each other due to the factor expressed by the parameter q (or ν).

Finally, the commutation relations for anyons of opposite types of nonlocality, i.e., of γ -type and δ -type, are to be exhibited (\mathbf{x}, \mathbf{y} arbitrary):

$$\{a_i(\mathbf{x}_\gamma), a_j(\mathbf{y}_\delta)\} = 0, \quad (14)$$

$$\{a_i(\mathbf{x}_\gamma), a_j^\dagger(\mathbf{y}_\delta)\} = 0, \quad (15)$$

$$\{a_i(\mathbf{x}_\gamma), a_j^\dagger(\mathbf{x}_\delta)\} = \delta_{ij} q \left[\sum_{\mathbf{y} < \mathbf{x}} - \sum_{\mathbf{y} > \mathbf{x}} \right] c_i^\dagger(\mathbf{y}) c_i(\mathbf{y}), \quad (16)$$

as well as the relations that result from all these (i.e., (10)–(16)) by applying Hermitian conjugation.

As proven in [4]–[5], the set of N anyons defined in (9) through the formulae

$$\begin{aligned} E_j^+ &\equiv I_{j,j+1} = \sum_{x \in \Omega} I_{j,j+1}(\mathbf{x}), & E_j^- &\equiv I_{j+1,j} = \sum_{x \in \Omega} I_{j+1,j}(\mathbf{x}), \\ H_j &\equiv I_{jj} - I_{j+1,j+1} = \sum_{x \in \Omega} \{I_{jj}(\mathbf{x}) - I_{j+1,j+1}(\mathbf{x})\}, \end{aligned}$$

where

$$\begin{aligned} I_{j,j+1}(\mathbf{x}) &= a_j^\dagger(\mathbf{x}_\gamma) a_{j+1}(\mathbf{x}_\gamma), & I_{j+1,j}(\mathbf{x}) &= a_{j+1}^\dagger(\mathbf{x}_\delta) a_j(\mathbf{x}_\delta), \\ I_{jj}(\mathbf{x}) &= a_j^\dagger(\mathbf{x}_\alpha) a_j(\mathbf{x}_\alpha) = N_j(\mathbf{x}) \end{aligned}$$

(here α is γ or δ ; $N_j = c_j^\dagger c_j$), supplies a (bilinear) realization of the $U_q(sl_N)$ algebra.

3. To realize analogously the algebra $U_{q;s_1,s_2,\dots,s_{n-1}}(gl_n)$, we have to use a modified (with respect to standard definition (9) and to the relations (11)–(16)) definition of transmuted (from the fermionic prototypes (6)–(7)) oscillators, namely

$$\begin{aligned} A_k(\mathbf{x}_\gamma) &= \exp\left(i\nu \sum_{\mathbf{y} \neq \mathbf{x}} \Theta_{\gamma\mathbf{x}}(\mathbf{x}, \mathbf{y}) N_k(\mathbf{y})\right) \prod_{j=1}^{k-1} s_j^{\mathbf{y}}^{\sum N_j(\mathbf{y})} c_k(\mathbf{x}), \\ A_k(\mathbf{x}_\delta) &= \prod_{j=k}^{n-1} s_j^{\mathbf{y}}^{\sum N_{j+1}(\mathbf{y})} \exp\left(i\nu \sum_{\mathbf{y} \neq \mathbf{x}} \Theta_{\delta\mathbf{x}}(\mathbf{x}, \mathbf{y}) N_k(\mathbf{y})\right) c_k(\mathbf{x}). \end{aligned} \quad (17)$$

We'll call them the "quasi-anyonic" operators, since under restrictions (5) and $s=1$ they turn into usual anyonic ones, see eq. (9). Below, let $s_j = \exp(i\pi\rho_j)$.

By direct examination, one verifies the following. For the coinciding modes of the operators $A_i(\mathbf{x}_\alpha)$ and the same cuts, the relations of commutation remain the same as those in eqs. (11)–(13). For those with opposite cuts (γ and δ) and at coinciding modes, one has

$$\{A_k(\mathbf{x}_\gamma), A_k^\dagger(\mathbf{x}_\delta)\} = q^{\sum_{\mathbf{y} \neq \mathbf{x}} \text{sgn}(\mathbf{x}-\mathbf{y}) N_k(\mathbf{y})} \prod_{j=1}^{k-1} s_j^{\mathbf{y}}^{\sum N_j(\mathbf{y})} \prod_{j'=k}^{n-1} s_{j'}^{\mathbf{y}}^{-\sum N_{j'+1}(\mathbf{y})}. \quad (18)$$

For different modes and the same cut γ , we obtain

$$\begin{aligned} A_i(\mathbf{x}_\gamma) A_j(\mathbf{y}_\gamma) + s_j^{-\text{sgn}(i-j)} A_j(\mathbf{y}_\gamma) A_i(\mathbf{x}_\gamma) &= 0, \\ A_i(\mathbf{x}_\gamma) A_j^\dagger(\mathbf{y}_\gamma) + s_j^{\text{sgn}(i-j)} A_j^\dagger(\mathbf{y}_\gamma) A_i(\mathbf{x}_\gamma) &= 0, \end{aligned} \quad (19)$$

while for the same cut δ we have

$$\begin{aligned} A_i(\mathbf{x}_\delta) A_j(\mathbf{y}_\delta) + s_{i-1}^{\text{sgn}(i-j)} A_j(\mathbf{y}_\delta) A_i(\mathbf{x}_\delta) &= 0, \\ A_i(\mathbf{x}_\delta) A_j^\dagger(\mathbf{y}_\delta) + s_{i-1}^{-\text{sgn}(i-j)} A_j^\dagger(\mathbf{y}_\delta) A_i(\mathbf{x}_\delta) &= 0. \end{aligned} \quad (20)$$

Finally, for opposite cuts, the relations of permutation are, for $\forall x, y$,

$$\{A_i(\mathbf{x}_\gamma), A_j(\mathbf{y}_\delta)\} = 0, \quad i \leq j, \quad \{A_i(\mathbf{x}_\gamma), A_j^\dagger(\mathbf{y}_\delta)\} = 0, \quad i < j, \quad (21)$$

and, again for arbitrary x, y ,

$$A_i(\mathbf{x}_\gamma)A_j(\mathbf{y}_\delta) + s_j^{-1}s_{i-1}A_j(\mathbf{y}_\delta)A_i(\mathbf{x}_\gamma) = 0, \quad i > j, \quad (22)$$

$$A_i(\mathbf{x}_\gamma)A_j^\dagger(\mathbf{y}_\delta) + s_j s_{i-1}^{-1}A_j^\dagger(\mathbf{y}_\delta)A_i(\mathbf{x}_\gamma) = 0, \quad i > j. \quad (23)$$

To the considered relations for quasianyons, also their Hermitian counterparts are to be added. It is obvious that at $s_1 = s_2 = \dots = s_{n-1} = 1$ all the relations of permutation for the quasianyonic operators $A_i(\mathbf{x}_\alpha)$ go over into those for usual anyons.

It can be shown by direct verification that the following assertion is true.

Proposition. The generators I_{jj+1}, I_{j+1j} , $j = 1, 2, \dots, n-1$, and I_{ii} , $i = 1, 2, \dots, n$, realized in the form

$$I_{k,k+1} = \sum_{x \in \Omega} I_{k,k+1}(\mathbf{x}), \quad I_{k+1,k} = \sum_{x \in \Omega} I_{k+1,k}(\mathbf{x}), \quad I_{kk} = \sum_{x \in \Omega} I_{kk}(\mathbf{x}),$$

with local densities taken as

$$I_{k,k+1}(\mathbf{x}) = A_k^\dagger(\mathbf{x}_\gamma)A_{k+1}(\mathbf{x}_\gamma), \quad I_{k+1,k}(\mathbf{x}) = A_{k+1}^\dagger(\mathbf{x}_\delta)A_k(\mathbf{x}_\delta),$$

$$I_{kk}(\mathbf{x}) = A_k^\dagger(\mathbf{x}_\alpha)A_k(\mathbf{x}_\alpha) = N_k(\mathbf{x}),$$

close into the (global bi-)algebra $U_{q;s_1,s_2,\dots,s_{n-1}}(gl_n)$ defined by the relations (1)–(4).

4. Let us make some concluding remarks. The formulas presented in previous section describe the set of generalized 'anyons', or quasi-anyons, which possess maximum of possible inter-mode dependences. Not only the coinciding modes of our quasi-anyons feel each other (with braiding characterized by the q) at distinct sites of the lattice (this property reproduces that of usual anyons, see eqns. (11)–(12) and their conjugates) but moreover, as exhibit relations (19)–(23), the quasi-anyons participate in *graded braiding relations* that depend on the values of indices, thus realizing a kind of ordering within the set of quasi-anyonic species.

The system of quasi-anyons given by eq. (17) differs from the modified anyons used in [12], as shows direct comparison at $n = 2$. Multimode anyon-like deformed oscillators proposed in refs. [7], [13] (although being not nonlocal objects and not tied to a specific dimension) resemble our quasi-anyons of Section 3, and it is useful to analyze the (dis)similarities in more detail. Finally, there is an interesting issue concerning the alternative: to attribute the (s_j -dependent) modifying factors in eq. (17) either to a change of a disorder operator (e.g., in the spirit of ref.[14]), or to a (multiparameter) deformation of the multimode fermionic oscillator.

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