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Symmetries of the Classical Integrable Systems and 2-Dimensional Quantum Gravity: a 'Map'

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Abstract

We draw attention to the connections recently established by others between the classical integrable KdV and KP hierarchies in $1+1$ and $2+1$ dimensions respectively and the matrix models which relate to the partition functions of 2-dimensional $(1+1$ dimensional) quantum gravity. The symmetries of the classical KP hierarchy in $2+1$ dimensions are fundamental to this connection.

1 Introduction and background: the Symposium "KdV '95" and the 'map' Solitons

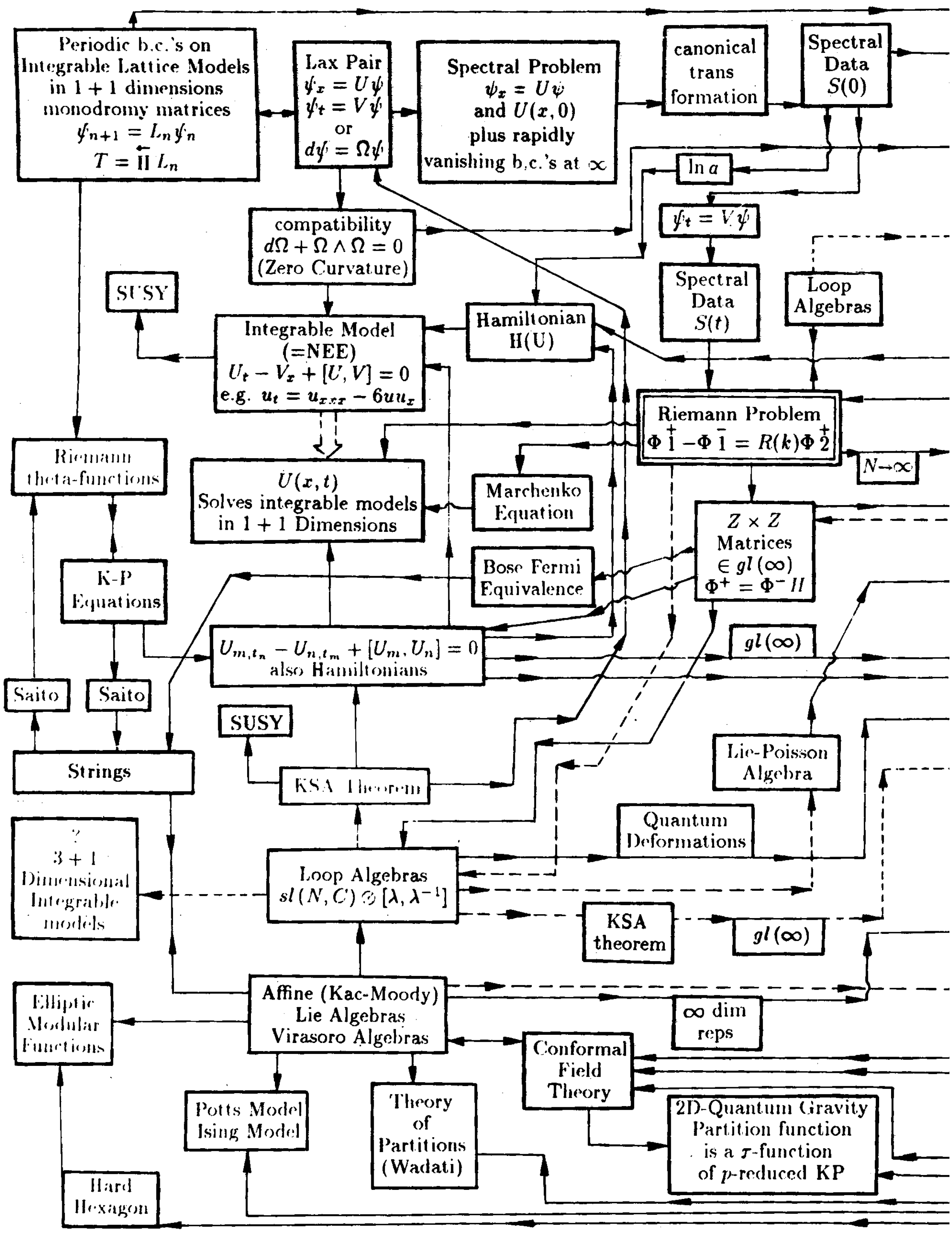
Circumstances have prevented either of the two authors from taking part in this meeting 'Symmetries in nonlinear mathematical physics' and we send this short contribution for the Vol.4 of the Proceedings in the form of a 'map', the Fig.1, called 'SOLITONS'. This map has already appeared in different forms in a number of different places (see the Ref. [53] listed in §4) and it first appeared in its present, and so far 'final', form in [7,8] (see Refs. [7, 8] listed in §4). It has subsequently appeared in this 'final' form in Ref. [53] and then in our contribution [87] to the Proceedings of the International Symposium held in Amsterdam, April 23–26, 1995, called KdV '95. The meeting KdV '95 commemorated the centennial of the publication of the famous paper [1] by D.J. Korteweg and G. de Vries of 1895.

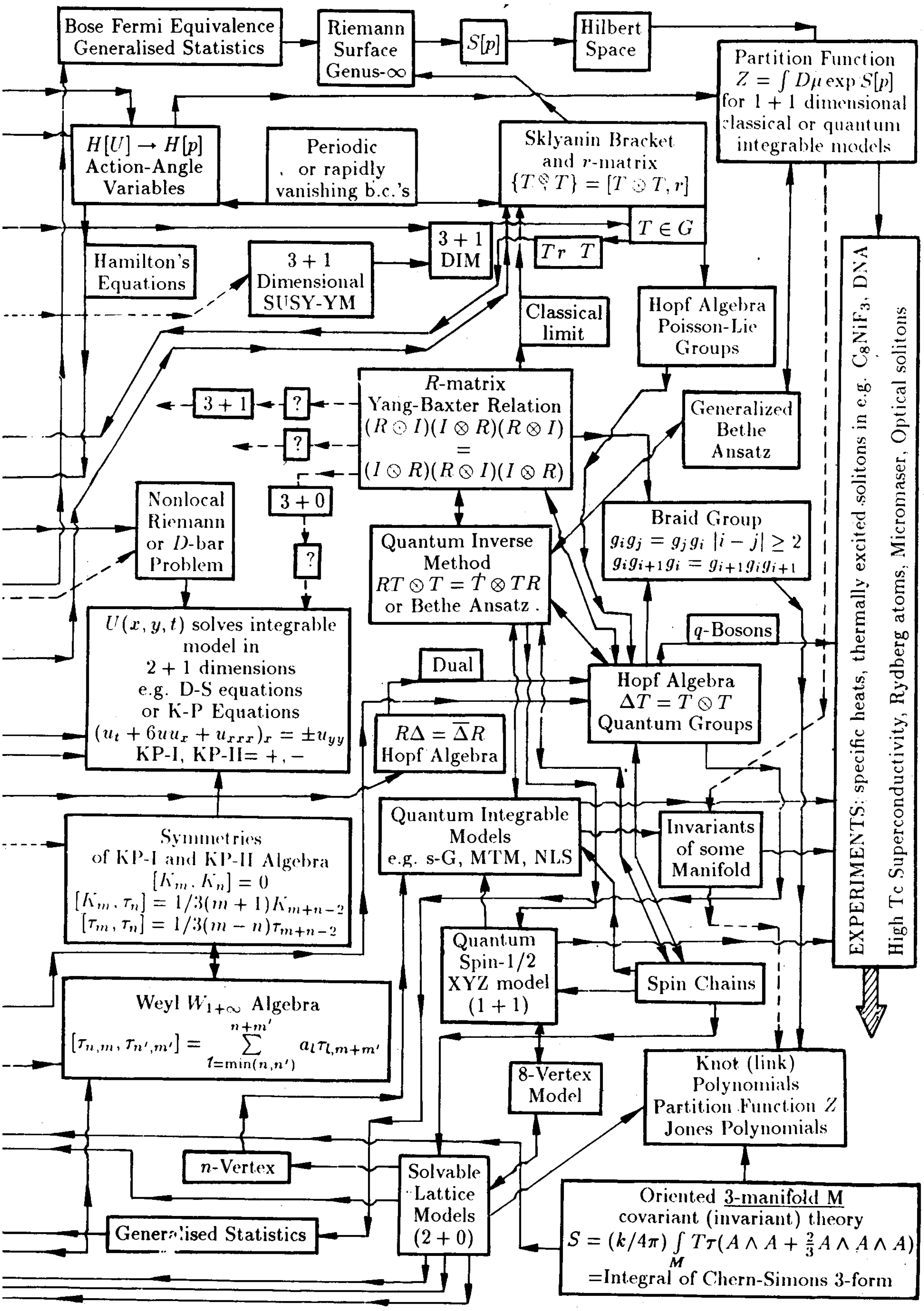
The Fig.1 as it is here reproduced now displays the captions as displayed in [87], and the reference numbers there refer to the reference list taken from [87] and attached here again as the §4 which follows below. In the meantime a small change is also included: 'Optical solitons' is added to the EXPERIMENTS 'box' at the extreme right of the Fig.1 and *this* 'final' form, reproduced as the Fig.1 again now, was first published in [88]. Then again a connecting arrow between the 'box' third column from the right called 'Quantum Integrable Models e.g. s-G, MTM, NLS', where NLS is the nonlinear Schrödinger equation, is (for NLS) directed at this 'optical solitons', and the symbol NLS in this box is also new compared with Ref. [87] (which reads 'e.g. s-G, MTM' only).

Fig.1 Overview of generalised 'Soliton' theory as of August 1991 taken from Refs. [7, 8]. A hard arrow indicates minimal connection (at least) between the boxes is already established and most hard arrows are actual mappings. Dashed arrows indicate expectation by the authors that some such minimal connection can be achieved or stronger. Note how p-reduction of the KP equations reaches the string equation for 2D-Quantum Gravity coupled to (p, q) conformal matter [72, 81]. Pure quantum gravity is $p = 2$ the case considered by Migdal [50]. $\Longrightarrow$

"SOLITONS"

U, V and Ψ are $N \times N$ Matrices $\left\{ \begin{matrix} sl(N,C) \\ gl(N,C) \end{matrix} \right\} = g \ni U, V$




Bose Fermi Equivalence
Generalised Statistics
Riemann Surface Genus-∞
S[p]
Hilbert Space
Partition Function
Z = ∫ Dμ exp S[p]
for 1 + 1 dimensional classical or quantum integrable models
H[U] → H[p]
Action-Angle Variables
Periodic , or rapidly vanishing b.c.'s
Sklyanin Bracket and r-matrix
T ∈ G
3 + 1 DIM
Tr T
Hamilton's Equations
3 + 1 Dimensional SUSY-YM
Classical limit
Hopf Algebra Poisson-Lie Groups
R-matrix
Yang-Baxter Relation
3 + 1
?
Generalized Bethe Ansatz
3 + 0
Nonlocal Riemann or D-bar Problem
Quantum Inverse Method
RT ⊗ T = T ⊗ TR
or Bethe Ansatz .
Braid Group
q-Bosons
U(x, y, t) solves integrable model in 2 + 1 dimensions
e.g. D-S equations
or K-P Equations
KP-I, KP-II= +, −
Dual
Hopf Algebra
ΔT = T ⊗ T
Quantum Groups
RΔ = Δ̄R
Hopf Algebra
EXPERIMENTS: specific heats, thermally excited solitons in e.g. C8NiF3, DNA
High Tc Superconductivity, Rydberg atoms, Micromaser, Optical solitons
Quantum Integrable Models
e.g. s-G, MTM, NLS
Invariants of some Manifold
Symmetries of KP-I and KP-II Algebra
Quantum Spin-1/2 XYZ model (1 + 1)
Spin Chains
Weyl W1+∞ Algebra
8-Vertex Model
Knot (link) Polynomials
Partition Function Z
Jones Polynomials
n-Vertex
Solvable Lattice Models (2 + 0)
Oriented 3-manifold M
covariant (invariant) theory
=Integral of Chern-Simons 3-form
Generalised Statistics

We think 'optical solitons' as such was first used in [86] so that (see [86]) the connecting arrow is also from s-G (the sine-Gordon equation) to 'optical solitons'. But s-G also connects to 'High T_c superconductivity' as well as to the ferromagnetic chain $CsNiF_3$. Finally MTM means, of course, the quantum massive Thirring model [53] and this is fermi-bose equivalent [53] to the quantum s-G model.

2 The symmetries of the 'map'

The relevance of the Fig.1 to the present meeting is, apart from the well known relevance of symmetries to integrable systems and solitons anyway (completely integrable Hamiltonian systems necessarily have such symmetries), precisely the set of connections marked in the figure in the fourth column from the right between the three vertically connected 'boxes' there. The contents of each of these three 'boxes' read respectively

Symmetries of KP-I and KP-II Algebra

$$\begin{aligned} [K_m, K_n] &= 0 \\ [K_m, \tau_n] &= \tfrac{1}{3}(m+1)K_{m+n-2} \quad , \\ [\tau_m, \tau_n] &= \tfrac{1}{3}(m-n)\tau_{m+n-2} \end{aligned} \tag{1}$$

$\mathrm{Weyl}_{1+\infty}$ Algebra

$$[\tau_{n,m}, \tau_{n',m'}] = \sum_{\ell=\min(n,n')} a_\ell \tau_{\ell,m+m'} \quad , \tag{2}$$

---

and

2D Quantum Gravity
Partition function is a (3)
τ − function of p-reduced KP .

A representation of the algebra of the symmetries of KP-I and KP-II displayed in the expressions (1) is derived in [83], and the K_n are the *isospectral* symmetries of KP: 'KP' means the Kadomtsev–Petviashvili equations and these are quoted on the 'map', as in [83], as

$$(u_t + 6uu_x + u_{xxx})_x = \pm u_{yy} \tag{4}$$

in the 'box' directly above the 'box' (1). The τ_n in the expressions (1) are then directly related [83] to certain non-isospectral symmetries: the scaling $\frac{1}{3}$, and the form of the Virasoro algebra [70, 87] $[\tau_m, \tau_n] = \frac{1}{3}(m-n)\tau_{n+m-2}$ arise from the details of the scaling of KP as adopted in (4) and [83] and the details of the consequent symmetries K_n, τ_n as they are treated in [83].

Note that all of the symmetries K_n and τ_n in $2+1$ dimensions are of the general Lie–Bäcklund type – in some contrast with e.g. the Lie symmetries studied in depth in Ref. [89] at this meeting. Consequently in the context of 2-dimensional quantum gravity (§3 next and the expressions (3) above) one can introduce the so-called "p-reduced" KP-I flows

[72, 81] which are constrained [7] by Douglas's 'string equation' so that the consequent infinite set of constraints satisfies the Weyl W_∞ algebra marked in Fig.1 (in the 'box' with contents (2)) in the form

$$\left[\tau_{n,m}, \tau_{n',m'}\right] = \sum_{\ell=\min(n,n')}^{n+n'} a_\ell \tau_{\ell,m+m'}. \tag{5}$$

In eqns. (5) the non-isospectral symmetries τ_m can be identified as the $\tau_{0,m}$. With $n \geq -1$ and $n+m \geq -1$ the W_∞ algebra is isomorphic to that generated by $\tau_{n,m} = x^{n+m+1} d^{n+1}/dx^{n+1}$ and it excludes the commuting isospectral flows identified in expressions (1) as the symmetries K_m. Thus in the same context of quantum gravity [81] the W_∞ algebra has been extended to the $W_{1+\infty}$ algebra through generalised flows

$$\partial\tau/\partial t_{-1,\ell} = C_{-1,\ell}\tau, \quad \partial\tau/\partial t^*_{n,m} = C_{n,m}\tau = 0, \qquad n \geq 0. \tag{6}$$

The $t_{-1,\ell}$ are times t_ℓ of the KP hierarchy and the $t^*_{n,m}$ are defined in [81]. The $C_{n,m}$ so introduced now satisfy the $W_{1+\infty}$ algebra

$$\begin{aligned} [C_{-1,n}, C_{-1,m}] &= 0, \ \ [C_{-1,n}, C_{0,m}] = nC_{-1,n+m}, \quad [C_{0,n}, C_{0,m}] = (n-m)C_{0,n+m}, \\ [C_{0,n}, C_{1,m}] &= (m-2n)C_{1,n+m} + n(n+1)C_{0,n+m}, \ldots \end{aligned} \tag{7}$$

(namely the eqns. (8) of [81]). In this form the symmetries K_n of Ref. [83] can be identified as the $C_{-1,n}$, and the flows on the left of eqn. (6) are the KP flows; the symmetries τ_n are the $C_{0,n}$ and the $\tau_n \equiv C_{0,n}$ satisfy the Virasoro algebra in proper form – with however (see §3) the central charge of this algebra set to zero. Note that the $C_{0,m}$ enter as constraints in the generalised flows appearing in the equations to the right in (6) and there set to zero.

It is the Ref. [7] by one of us which gives the brief connection to '2D Quantum Gravity' as this is marked on the Fig.1 namely in the connected 'boxes' with contents the expressions (2) and (3) above. We also draw attention to Ref. [82]. However, Refs. [7, 81-83] are in themselves scarcely sufficient to take any but the most expert readers from the symmetries of KP-I to 2D-quantum gravity. So in sending the 'map' the Fig.1 to the proceedings of this meeting on 'Symmetries in nonlinear mathematical physics' we also attach, as the §3 next, the §6 of the Ref. [87] essentially as it was published in the proceedings of KdV '95. Little change from the original §6 of [87] is made in this §3 next, but some errors are corrected[1‡] and where there has been back-reference to equations used in the earlier sections of [87] these equations are now inserted or the situation otherwise accomodated. The §4 following this §3 then reproduces the complete reference list of [87] (with some corrections) and so provides the reader with the opportunity, if he so wishes, to look again at the whole evolution of integrable systems during 1834–1995. The new references [87], [88] and [89] are added to the reference list in §4 as compared with that of [87].

In this way the §3 which now follows is self-contained and is, we think, usefully appended in order to illustrate more completely the role of the symmetries of the KP-I hierarchy in current theories for the partition functions Z of 2D-quantum gravity coupled most generally [72] to '(p,q)-conformal matter'.

[1‡] As published [87] contained many printers errors. These are being rectified in a corrected format to appear in *Acta Applicandae Mathematicae* shortly.

Regretfully we do not have opportunity to elaborate still further here on this remarkable example of the application of symmetries in nonlinear mathematical physics.

3 KdV and 2-dimensional quantum gravity

In formulating this short summary we have been much influenced by the presentation in Ref. [67].

In contrast with the partition functions elaborated for example in our [53] we follow Polyakov [68] and write the partition function Z for 2-dimensional quantum gravity ($\hbar = 1$)[2‡]

$$Z = \sum_p \int \mathcal{D}g \int \mathcal{D}X \exp S[g, X; p], \tag{8}$$

$$S[g, X; p] = -\lambda_1 \int_{\sum_p} \sqrt{g} - \left(\frac{\lambda_2}{2\pi}\right) \int_{\sum_p} R\sqrt{g} - \int_{\sum_p} \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\mu. \tag{9}$$

The functional integral $\int \mathcal{D}g$ is integration over all possible metrics on the 2- surface $\sum_p$ of genus p and the summation is over such surfaces; $\int \mathcal{D}X$ is integration over all mappings $X : \sum_p \to \mathbf{R}^D$, where D denotes dimension (these mappings are the string fields) and R is the scalar curvature of the metric g. The constants $\lambda_1, \lambda_2 \in \mathbf{R}$ are the cosmological constant (λ_1) and the string coupling (λ_2). Pure 2-dimensional quantum gravity then has the partition function

$$Z = \sum_p \int \mathcal{D}g \exp S[g; p], \tag{10}$$

$$S[g; p] = -\lambda_1 \int_{\sum_p} \sqrt{g} - \left(\frac{\lambda_2}{2\pi}\right) \int_{\sum_p} R\sqrt{g}. \tag{11}$$

A problem now is to make sense of these two heuristically motivated functional integrals: a route has been [50, 69] to consider the triangulations of 2 surfaces and more generally [67] the covering of surfaces by squares, hexagons, etc. A calculation of the number of ways of doing this for each genus p can be used to approximate Z, eqns. (8), (9). This approximation is shown to be valid for each genus p for special values of the dimension D namely for

$$D = 1 - 6[m(m+1)]^{-1}, \quad m = 2, 3, \ldots \quad . \tag{12}$$

Experts may recognize in formula eqn. (12) one of the two conditions, this one on the central charge usually called c but here called D, that there must be for there to be a unitary representation of a Virasoro algebra [70]. The particular case $m = 2$ ($D = 0$) is the case of pure quantum gravity eqns. (10), (11). More generally one finds the representation for Z eqns. (8), (9) which is of the form [67]

$$Z = \sum_{p=0}^{\infty} \int Z_p(A) dA \simeq \sum_{p=0}^{\infty} n^{2-2p} \sum_{q=0}^{\infty} (-\lambda)^q W_{\{x\}}(p; q) \tag{13}$$

[2‡] We follow convention and omit the surface elements in the surface integrals.

with[3‡]

$$\lambda \to \lambda_c,\ n \to \infty;\ n(\lambda - \lambda_c)^{-\gamma} = 0(1), \tag{14}$$

and $\gamma = -1-(2m)^{-1}$ (so that $\gamma = -\frac{5}{4}$ for pure gravity $D = 0$). But this cannot define the functional integral eqns. (8), (9) for Z since the right side only has meaning asymptotically. However, from the known asymptotic behaviour of $W_{\{x\}}(p;q)$ as $q \to \infty$ one deduces

$$Z \equiv Z(t) = \sum_{p=0}^{\infty} t^{2-2p} b_p \Gamma(\gamma(2-2p)) + \text{reg. terms}, \tag{15}$$

where $b_p\{x\}$, $\gamma\{x\}$ for special choices of $m-2$ of the parameters[4‡] x such that $\gamma = -1-(2m)^{-1}$ (see above) help to determine the asymptotics of $W_{\{x\}}(p,q)$: parameters t in eqn. (16) are given by

$$t = n\left(1 - \frac{\lambda}{\lambda_c}\right)^{-\gamma} \tag{16}$$

and are renormalised string couplings with 'critical behaviours' as $\lambda \to \lambda_c$.

The asymptotic series eqn. (15) defines the perturbative theory for Z eqns. (8),(9). This series does not depend on the parameters $\{x\}$ mentioned[‡] but only on the integer m which determines D, eqn. (12) (and so determines an mth universality class). For a non-perturbative definition of the functional integral eqns. (8), (9) one needs a generating function for eqn. (13) and a candidate is

$$\ell n\ Z_n\left(\frac{1}{2}, \frac{\lambda}{4n}x_2, \frac{\lambda^2}{6n^2}x_3, ..., \frac{\lambda^{N-1}}{2Nn^{N-1}}x_N\right) \tag{17}$$

where $Z_n(t_1, t_2, ..., t_N)$ is the partition function of the *hermitian matrix model*

$$\begin{aligned} Z_n &= \int d\Phi \exp\{-TrU(\Phi)\}, \\ U(z)) &= \sum_{j=1}^{N} t_j z^{2j}. \end{aligned} \tag{18}$$

In this, Φ is an $n \times n$ hermitian matrix and

$$d\Phi = \prod_{i=1}^{n} d\Phi_{ii} \prod_{i<j} d\Phi_{ij} d\bar{\Phi}_{ij} \tag{19}$$

so that Z_n is a well defined finite dimensional integral. Of course this is a discretization for $n < \infty$ of the functional integral Z, and the matrix model eqn. (18) becomes non-trivial because one finds [67] that the dimension n of the integral, eqn. (18), is $n \to \infty$ within a certain double scaling limit which is the limit eqn. (21). First we rescale as

$$\Phi \to \beta^{\frac{1}{2}}\Phi,\quad \frac{\lambda\beta}{n} = 1,\quad x_j/2j = q_j \tag{20}$$

[3‡]For each p, $\lambda \equiv -e^{-\lambda_1 \epsilon}$ and λ_1 renormalises as $\lambda_1 = c\epsilon^{-1} + \lambda_1^0$ with $\epsilon \to 0$: λ_c as used in eqns. (14),(16) is thus $\lambda_c = -e^{-c}$.

[4‡]The parameters $\{x\}$ play a purely auxillary role.

(so the action $U(\Phi)$ gains a factor β). Then the problem is to evaluate $\ell n\ Z_n(\frac{1}{2}\beta, \beta q_2, ...$ $..., \beta q_N)$ under the limit

$$\beta = C_1 h^{-4-2m^{-1}}, \quad n\beta^{-1} = C_2 + C_1^{-1} h^4 \xi, \quad h \to 0 \tag{21}$$

in which

$$C_2 \equiv \lambda_c, \quad \lambda - \lambda_c \equiv C_1^{-1} h^4 \xi, \quad \xi \equiv -t^{2m(2m+1)-1} (C_1 \lambda_c)^{(2m+1)^{-1}}. \tag{22}$$

The equivalence between Z eqns. (8), (9) and matrix models eqn. (18) in this double scaling limit has sparked many papers which simply start from a matrix model *per se* (e.g. Refs. [49, 50], [71–73]). One can reduce the (finite dimensional) matrix model eqn. (18) to the expression [49, 67]

$$Z_n = \text{const} \prod_{i=1}^{n} h_{i-1}(t_1, ..., t_N) \tag{23}$$

where the h_n are normalisation constants of corresponding orthogonal polynomials $P_n(z)$ of degree n with leading coefficient unity taken against the weighting function $\exp -U(z)$: $h_n \delta_{nm} = \int_{-\infty}^{\infty} P_n(z) P_m(z) \exp - \sum_{j=1}^{N} t_j z^{2j}$. These h_n allow the introduction of $w_n = 4h_n/$ $/h_{n-1}$, $n = 1, 2, ...$, and these w_n satisfy a difference equation called the 'discrete string equation' studied in Ref. [67]. This system is an integrable system which is compatible with the integrable so-called Volterra hierarachy [67, 74].

For our purposes the connection now is that with

$$\begin{aligned} &\beta = C_1 h^{-5}, \quad n\beta^{-1} = C_2 + C_1^{-1} h^4 \xi, \\ &w_n \sim \rho(1 - 2h^2 u(\xi)), \quad h \to 0, \end{aligned} \tag{24}$$

which incorporates the limit eqn. (21) for the case $m = 2$, then, for an appropriate choice of the constants C_1, C_2, ρ, $u(\xi)$ satisfies the Painlevé I equation in the form

$$u_{\xi\xi} = 6u^2 + \xi. \tag{25}$$

More generally, that is for general m, the discrete string equation must be taken under the limit eqn. (21), and if $w_n \simeq \rho(1 - 2h^{4m^{-1}} u(\xi))$ then ρ can be chosen so that, after rescaling to eqn. (28) below (under which $\xi \to t$)

$$C_k R_k[u(t)] = t; \; C_k = \left[2\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)/\Gamma\left(k+\frac{1}{2}\right)\right](-1)^{k+1} \tag{26}$$

where $R_{k-1}[u(t)] = -\left(k - \frac{1}{2}\right)^{-1} \delta R_k[u(t)]/\delta u$ and (see eqn. (57) of [87]) defines the kth *stationary KdV flow* $R_{k-1}[u] = 0$ [87]. The $R_k[u]$ are introduced in [87] as coefficients of the resolvent of the Schrödinger operation eigenvalue equation just as this was done in Ref. [46].

In more detail Gross and Migdal [49] start from the matrix model

$$Z_n(\beta) = \int d\Phi \exp[-\beta Tr U(\Phi)] \tag{27}$$

(which scales in the β) and see that

$$\ell n Z_n(\beta) = \text{reg. terms } - F(t) \tag{28}$$

in which, now, $t = (\beta - n)\beta^{-(2k+1)^{-1}}$ (compare eqn. (16)). Then the 'specific heat' is $f(t) = \ddot{F}(t)$ and for the kth 'multicritical point' this is given by the eqns. (29) which are:

$$\begin{aligned}
k = 1 \quad &: \quad t = f; \quad k = 2 : t = f^2 - \frac{1}{3}f'';\\
k = 3 \quad &: \quad t = f^3 - ff'' - \frac{1}{2}(f')^2 + \frac{1}{30}f'''';\\
k = 4 \quad &: \quad t = f^4 - 2f(f')^2 - 2f^2f'' + \frac{3}{5}(f'')^2 + \frac{4}{5}(f'f'') + \frac{2}{5}ff'''' - \frac{1}{35}f'''''';\\
k = 5 \quad &: \quad .... \qquad .
\end{aligned} \tag{29}$$

Eqns. (29) appear as eqns. (60) and (96) in [87] and they can be put in the form of eqn. (61) of [87]which is

$$[\hat{L}_n, \hat{A}_{k+1}] = 1 \tag{30}$$

where $\hat{L}_n = -d^2/d\xi^2 + u(\xi)$, the Schrödinger operator, while $\hat{A}_{k+1}$ is the A-operator forming the Lax pair for the $k+1$th KdV equation. Then Painlevé I, eqn. (25) is $k = 1$ which means $u_{\xi\xi\xi} - 12uu_\xi = 1$ and corresponds to the KdV equation with that scaling.

These rather remarkable results place the KdV hierarchy in an exceptional position vis-a-vis quantum gravity. It seems natural to look for the origin of this exceptional position in the *conformal invariance* of the KdV hierarchy. This raises questions about the representations of the Virasoro algebra in relation to 2-dimensional quantum gravity. A preferred candidate might have been the covariant Liouville equation

$$u_{xx} - u_{tt} = e^{\alpha u} \tag{31}$$

which is conformally invariant and plays a fundamental role already in string theory [75].

Conformally invariant field theories induce representations of two identical copies of the Virasoro algebra

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{1}{12}cm(m^2 - 1)\delta_{m,-n} \tag{32}$$

with $m, n \in \mathbf{Z}$ and c the central charge. With c identified as the dimension D at eqn. (12), we have already seen one (apparent) connection with the Virasoro algebra for which $0 \leq c < 1$: we sketch next an alternative theory of 2-dimensional quantum gravity where this algebra becomes immediately evident with however $c = 1$.[5‡] We refer now predominantly to the exposition by Fukuma, Kawai and Nakayama [72].

By starting from the one matrix ϕ^4-model which has action S scaled to $S = nTrU(\phi)$ with ϕ an $n \times n$ Hermitian matrix and $U(z) = \frac{1}{2}z^2 - \frac{1}{4}\lambda z^4$, as is already contained with the different scaling in eqn. (18), these authors find Schwinger–Dyson equations for the partition function $Z = \lim_{n\to\infty} Z_n$ taken in a continuum limit involving $a = n^{-\frac{2}{5}} \to 0$, an analogue of the limit eqns. (24). We shall now use n for the integers $n = -1, 0, 1, 2, ...$

[5‡]The meanings of these different choices for the central charge c, which include $c = 0$ (see §2), remain obscure to us at present.

following the notation of Ref. [72] (N is used for our *dimension* n in the Ref. [72]). The authors find there is a tau-function (τ-function) τ related to Z as $\tau^2 = Z$. This τ-function satisfies, for the integers $n = -1, 0, 1, 2, ...$

$$L_n \tau = 0,$$

$$2L_n = \frac{1}{2} \sum_{p+q=-2n} pq x_p x_q + \sum_{p-q=2n} p x_p \frac{\partial}{\partial q} + \frac{1}{2} \sum_{p+q=2n} \partial_p \partial_q + \frac{1}{8} \delta_n, 0. \tag{33}$$

The p, q run over positive odd integers and the L_n satisfy a Virasoro algebra, eqn. (32) with $c = 1$.

The authors of Ref. [72] *conjecture*, but virtually prove by explicit calculation, that τ satisfying eqns. (33) is actually a τ-function [76] of the KdV hierarchy satisfying *in addition* the equation

$$L_{-1} \tau = 0. \tag{34}$$

The conjectured equivalence of the two statements that $L_n \tau = 0$, eqns. (33), and that τ is a τ-function of the KdV hierarchy[6‡] is strongly supported by the explicit calculation which reaches

$$x_1 + (3x_3 \partial_1^2 \ell n \tau + 5x_5 \partial_1 \partial_3 \ell n \tau + ...) = 0 \tag{35}$$

(eqn. (3.6) of Ref. [72]) for the equation $L_{-1}\tau = 0$ differentiated once with respect to x_1. The many variables $x_1, x_3, x_5, ...$ arise in the following way. It is necessary to go to the KP hierarchy [76] of which the first member is (up to scaling) the KP-I equation in (2+1)-dimensions, eqn. (4) above. The KP-I hierarchy has a τ-function [76] and τ satisfies [72,76]

$$\partial_1 \partial_m \ell n \tau = (L^m)_{-1}, \tag{36}$$

where L is the pseudo-differential operator $L = \partial + u_2 \partial^{-1} + ...$, essentially the square root $\hat{R}$ of $-\hat{L}_u$, $\hat{L}_u \equiv \dfrac{-\partial^2}{\partial x^2} + u(x)$, the Schrödinger operator introduced in §4 of [87], and ∂ is $\partial_{x_1} \equiv \partial_1$, while $(\quad)_{-1}$ stands for the coefficient of ∂^{-1}. The KP-I equations in $(2+1)$-dimensions contain by reduction many of the integrable systems in $(1+1)$-dimensions [76], and in particular the condition of 'two- reduction'[7‡] which is $(L^2)_- = 0$, where $(\quad)_-$ stands for the negative-powers-in-∂-part, restricts to the variables $x_1, x_3, x_5....$ and yields

$$(L^{2k-1})_{-1} = 2R_k[-2u_2], \qquad k \geq 1 \tag{37}$$

where R_k is again the coefficient R_k of the resolvent of the Schrödinger operator eigenvalue equation $\hat{L}_u \phi = -\lambda \phi$, and determines both the stationary KdV flows $R_{k-1}[u] = 0$ and the eqns. (29). By combining eqns. (35), (36) and (37) the authors of Ref. [72] then reach

$$\frac{1}{2} x_1 + \sum_{k=1}^{\infty} (2k+1) x_{2k+1} R_k[-2u_2] = 0 \tag{38}$$

[6‡]The τ-function as described in Ref. [76] can be traced to the work of Hirota [77] and ultimately to his solution of the KdV equation [78]. Caudrey et al. [79, 80] gave the comparable solution of the sine-Gordon equation soon after.

[7‡]The case of 2-reduction is thought of as pure quantum gravity but includes all of the multicritical points of Refs. [49,50]: more generally gravity is coupled to "(p, q) conformal matter" [72, 81, 82].

from which one picks off eqns. (29), up to scaling, by setting all but one of the x_{2k+1} to zero ($R_k[-2u_2]$ depends only on the variable x_1). Thus, recalling the differentiation with respect to x_1, the equation $L_{-1}\tau = 0$ is a once integrated version of eqn. (38) under the assumption that τ is a τ-function of the KdV hierarchy. From this the other equations $L_n\tau = 0$ $(n \geq 0)$ should follow from $L_{-1}\tau = 0$; and as an immediate check, with all x's zero except x_1, x_5, eqn. (38) is of course the Painlevé I equation $f^2 + \frac{1}{3}\partial_1^2 f = x_1$ for $f = 2\partial_1^2 \ell n\tau = 2u_2$. Then using this and eqn. (36) one finds [72] that $L_0\tau = 0,\ L_1\tau = 0$, at least.

From this result one can guess (as in Ref. [72]) that the "3-reduction" of the KP-hierarchy $(L^3)_- = 0$, which will produce the Boussinesq hierarchy whose first member, up to scaling is [87]

$$u_{tt} = u_{xx} + \left[\frac{3}{2}u^2 + u_{xx}\right]_{xx}, \tag{39}$$

is equivalent to the Schwinger-Dyson equations

$$L_n\tau = 0$$

$$3L_n = \frac{1}{2}\sum_{p+q=-3n} pqx_px_q + \sum_p px_p\partial_{p+3n} + \frac{1}{2}\sum_{p+q=3n}\partial_p\partial_q + \frac{1}{3}\delta_{n,0}, \tag{40}$$

and the L_n again satisfy the Virasoro algebra eqns. (32) with $c = 1$.

A new feature is that τ is not now uniquely determined by eqns. (40). Much as in §2 the authors of Ref. [72] therefore add a large set of further equations which form a representation of the W-algebra [72, 81, 82]. The explicit expression for W_n chosen in Ref. [72] is

$$\begin{aligned} 3^{\frac{3}{2}}W_n \quad = \quad & \sum_{p+q+r=-3n} pqrx_px_qx_r + 3\sum_{p+q-r=-3n} pqx_px_q\partial_r + \\ & 3\sum_{p-q-r=-3n} px_p\partial_q\partial_r + \sum_{-p-q-r=-3n}\partial_p\partial_q\partial_r \end{aligned} \tag{41}$$

where $p,\ q,\ r$ run over positive integers $\not\equiv 0$ (mod. 3). And the total algebra becomes

$$\begin{aligned} [L_n, L_m] &= (m-n)L_{m+n} + \tfrac{1}{12}m(m^2-1)\delta_{m,-n}, \\ [L_n, W_m] &= (2n-m)W_{n+m}, \\ [W_n, W_m] &= -\tfrac{1}{10}\delta_{n+m,0}n(n^2-1)(n^2-4) + \\ & (n-m)\left[\tfrac{3}{2}(n^2+4nm+m^2) + \tfrac{27}{2}(n+m) + 21\right]L_{n+m} - \\ & 9(n-m)U_{n+m}, \\ \text{and } U_n &= \Sigma_{k\leq -2}L_kL_{n-k} + \Sigma_{k\geq -1}L_{n-1}L_k. \end{aligned} \tag{42}$$

It is worthwhile comparing this W algebra with the W_∞ algebra of eqns. (5) and with the corresponding expressions in the $W_{1+\infty}$ algebra of eqns. (7) for which the correspondences do not seem to be complete. Still the equations

$$\begin{aligned} L_n\tau = 0 \quad &, \quad n = -1, 0, 1, ..., \\ W_n\tau = 0 \quad &, \quad n = -2, -1, 0, ... \end{aligned} \tag{43}$$

prove to form a closed and consistent set, and it is conjectured, in analogy with the case of 2- reduction and the KdV hierarchy, that an equivalent statement to τ is a τ-function of the KdV hierarchy with $L_{-1}\tau = 0$ is now τ is a τ-function of the *Boussinesq* hierarchy together with, in formal terms, the same additional condition

$$L_{-}\tau = 0. \tag{44}$$

As described already in §2, by adjoining a representation of the $U(1)$ Kac-Moody Lie algebra, the whole problem has been transformed to the flows eqns. (6) with the $W_{1+\infty}$ algebra eqns. (7) for the $C_{n,m}$. These results explain the connections between the 'boxes' marked on the 'map', the Fig. 1, with contents the expressions (1), (2) and (3) of §2 (Note that for compactness the W_∞ algebra eqns. (5) is actually shown in the expressions (2) not the $W_{1+\infty}$ algebra (7), while the τ-function is actually [72] the *square root* of the partition function in the expressions (3)).

In this way we can see the fundamental importance of the symmetries of the KP-I hierarchy, together with their generalisations to the larger algebras arising through constraints, to the theory of 2-dimensional quantum gravity coupled to (p, q) conformal matter. As noted in Ref. [72] the general framework is by no means complete. We have tried to make this plain also in this present sketch, taken from Ref. [87]. For, to our understanding, a total unification in all of its detail of the different approaches in Refs. [81, 82], in [83], and in [72], and indeed in [67], is still to be achieved.

We mention finally that the Ref. [87] from which this §3 is drawn was concerned first of all with the KdV equation and its hierarchy in $1+1$ dimensions. The reader is referred to the end of the §6 of [87] for the further remarks on this topic. In essence these express the amazement of the two authors that J.S. Russell's work [2,3] of 1840 and 1844, as well as the work of KdV [1] of 1895, all on water waves in canals of finite depth, could ever lead to, among other things such as the optical solitons [86–88][8‡] mentioned in §1, the results on 2-dimensional quantum gravity as they have been very incompletely sketched in the §§2 and 3 of this short paper.

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