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# Symmetry Reduction and Exact Solutions of the Eikonal Equation

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**Abstract**

By means of splitting subgroups of the generalized Poincaré group $P(1,4)$, ansatzes which reduce the eikonal equation to differential equations with fewer independent variables have been constructed. The corresponding symmetry reduction has been done. By means of the solutions of the reduced equations some classes of exact solutions of the investigated equation have been presented.

The relativistic eikonal (the relativistic Hamilton-Jacobi) equation is fundamental in theoretical and mathematical physics. Here we consider the equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{1}$$

In [1] it has been shown that the maximal local (in sense of Lie) invariance group of the equation (1) is the conformal group $C(1,4)$ of the 5-dimensional Poincaré-Minkowski space with the metric

$$s^2 = x^{\mathrm{A}} x_{\mathrm{A}} \equiv g^{\mathrm{AB}} x_{\mathrm{A}} x_{\mathrm{B}} = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2,$$

where $\mathrm{A}, \mathrm{B} = 0, 1, \ldots, 4$; $x_4 \equiv u$; $g^{\mathrm{AB}} = g_{\mathrm{AB}} = \{1, -1, -1, -1, -1\}\delta_{\mathrm{AB}}$, $\delta_{\mathrm{AB}}$ is the Kronecker delta. By means of special ansatzes multiparameter families of exact solutions of the eikonal equation were constructed [1–4].

It is well-known that the conformal group $C(1,4)$ contains the generalized Poincaré group $P(1,4)$ as a subgroup. The group $P(1,4)$ is a group of rotations and translations of the 5-dimensional Poincaré-Minkowski space. For investigation of the equation (1) we have used splitting subgroups [5–7] of the group $P(1,4)$. Using invariants [8] of splitting subgroups of the group $P(1,4)$, we have constructed ansatzes which reduce the equation (1) to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. Using the solutions of the reduced equations, we have found some classes of exact solutions of the eikonal equation.

1. Below we present ansatzes which reduce the equation (1) to ordinary differential equations (ODE), and we list the ODEs obtained as well as some solutions of the eikonal equation.

1. $u = \varphi(\omega),\ \ \omega = x_3;\ \ (\varphi')^2 = -1;\ \ u = \imath\varepsilon x_3 + C,\ \ \varepsilon = \pm 1.$

2. $u = \varphi(\omega),\ \ \omega = x_1^2 + x_2^2;\ \ (\varphi')^2 \omega = -1/4;\ \ u = \imath\varepsilon \left(x_1^2 + x_2^2\right)^{1/2} + C,\ \ \varepsilon = \pm 1.$

3. $u = \varphi(\omega),\ \ \omega = x_1^2 + x_2^2 + x_3^2;\ \ (\varphi')^2 \omega = -1/4;\ \ u = \imath\varepsilon \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2} + C,\ \ \varepsilon = \pm 1.$

4. $u^2 = -\varphi(\omega) + x_0^2,\ \ \omega = x_3;\ \ (\varphi')^2 - 4\varphi = 0;\ \ u^2 = -\left(\varepsilon x_3 + C\right)^2 + x_0^2,\ \ \varepsilon = \pm 1.$

5. $u^2 = \varphi(\omega) - x_3^2,\ \ \omega = x_0;\ \ (\varphi')^2 - 4\varphi = 0;\ \ u^2 = \left(\varepsilon x_0 + C\right)^2 - x_3^2,\ \ \varepsilon = \pm 1.$

6. $u^2 = \varphi(\omega) - x_3^2,\ \ \omega = x_1^2 + x_2^2;\ \ (\varphi')^2 \omega + \varphi = 0;\ \ u^2 = \left[\imath\varepsilon \left(x_1^2 + x_2^2\right)^{1/2} + C\right]^2 - x_3^2,$ $\varepsilon = \pm 1.$

7. $u^2 = \varphi(\omega) + x_0^2 - x_3^2,\ \ \omega = x_2;\ \ (\varphi')^2 + 4\varphi = 0;\ \ u^2 = \left(\imath\varepsilon x_2 + C\right)^2 + x_0^2 - x_3^2,\ \ \varepsilon = \pm 1.$

8. $u = \varphi(\omega) - x_0,\ \ \omega = x_2;\ \ \varphi' = 0;$

9. $u = \varepsilon\left(\varphi(\omega) - x_0\right),\ \ \omega = x_1^2 + x_2^2;\ \ \varphi' = 0;\ \ \varepsilon = \pm 1.$

10. $u = \varepsilon\left(\varphi(\omega) - x_0\right),\ \ \omega = x_3;\ \ \varphi' = 0;\ \ \varepsilon = \pm 1.$

11. $u = \varepsilon\left(\varphi(\omega) - x_0\right),\ \ \omega = x_1^2 + x_2^2 + x_3^2;\ \ \varphi' = 0;\ \ \varepsilon = \pm 1.$

The ansatzes (8)–(11) reduce the equation (1) to the linear ODEs. The ansatzes (1)–(11) can be written in the following form:

$$h(u) = f(x) \cdot \varphi(\omega) + g(x), \tag{2}$$

where $h(x), f(x), g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = \omega(x)$ are one-dimensional invariants of splitting subgroups of the group $\mathrm{P}(1,4)$.

12. $2x_0\omega = -\varphi(\omega) + x_3^2,\ \ \omega = x_0 + u;\ \ \omega\varphi' - \varphi + \omega^2 = 0;$
$2x_0\left(x_0 + u\right) = \left(\left(x_0 + u\right) - C\right)\left(x_0 + u\right) + x_3^2;$

13. $2x_0\omega = -\varphi(\omega) + x_1^2 + x_2^2 + x_3^2,\ \ \omega = x_0 + u;\ \ \omega\varphi' - \varphi + \omega^2 = 0;$
$2x_0\left(x_0 + u\right) = \left(\left(x_0 + u\right) - C\right)\left(x_0 + u\right) + x_1^2 + x_2^2 + x_3^2;$

14. $2x_0\omega = -\varphi(\omega) + x_1^2 + x_2^2,\ \ \omega = x_0 + u;\ \ \omega\varphi' - \varphi + \omega^2 = 0;$
$2x_0\left(x_0 + u\right) = \left(\left(x_0 + u\right) - C\right)\left(x_0 + u\right) + x_1^2 + x_2^2;$

The ansatzes (12)–(14) reduce the equation (1) to the linear ODEs. The ansatzes (12)–(14) can be written in the following form:

$$h(\omega, x) = f(x) \cdot \varphi(\omega) + g(x), \tag{3}$$

where $h(\omega, x), f(x), g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = \omega(x)$ are one-dimensional invariants of the splitting subgroups of the group $\mathrm{P}(1,4)$.

15. $u = \exp\left(d \cdot \arcsin\dfrac{x_2}{\omega} - \varphi(\omega)\right) - x_0,\ \ \omega = \left(x_1^2 + x_2^2\right)^{1/2};$

$$\varphi' = \frac{\imath\varepsilon}{\omega},\ \ u = \frac{C}{\left(x_1^2 + x_2^2\right)^{1/2 \cdot \varepsilon\imath d}} \exp\left(d \cdot \arcsin\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right) - x_0,\ \ \varepsilon = \pm 1.$$

16. $\ln\left(x_0^2 - u^2\right) - 2e \arcsin\dfrac{x_2}{\omega} + 2\,\mathrm{arch}\,\dfrac{x_0}{\sqrt{x_0^2 - u^2}} = \varphi(\omega),$

$$\omega = \left(x_1^2 + x_2^2\right)^{1/2};\ \ (\varphi')^2 + \frac{4e^2}{\omega^2} = 0,$$

$$\ln\frac{(x_0^2-u^2)}{C\,(x_1^2+x_2^2)^{e\varepsilon\imath}}-2e\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}+2\,\mathrm{arch}\,\frac{x_0}{\sqrt{x_0^2-u^2}}=0;\ \ \varepsilon=\pm1.$$

17. $\ln\left(x_0^2-u^2\right)+2e\arcsin\dfrac{x_2}{\omega}-2\,\mathrm{arch}\,\dfrac{x_0}{\sqrt{x_0^2-u^2}}=\varphi(\omega),$

$$\omega=\left(x_1^2+x_2^2\right)^{1/2};\ \ (\varphi')^2+\frac{4e^2}{\omega^2}=0,$$

$$\ln\frac{(x_0^2-u^2)}{C\,(x_1^2+x_2^2)^{e\varepsilon\imath}}+2e\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}-2\,\mathrm{arch}\,\frac{x_0}{\sqrt{x_0^2-u^2}}=0;\ \ \varepsilon=\pm1.$$

2. Next we present ansatzes which reduce the equation (1) to two-dimensional partial differential equations (PDE) and we list the PDEs obtained.

1. $u=\varphi(\omega_1,\omega_2),\ \omega_1=\left(x_1^2+x_2^2\right)^{1/2},\ \omega_2=x_3;\ \varphi_1^2+\varphi_2^2=-1,\ \varphi_i\equiv\dfrac{\partial\varphi}{\partial\omega_i}$ $(i=1,2)$.
2. $u=\varphi(\omega_1,\omega_2),\ \omega_1=x_0,\ \omega_2=x_3;\ \varphi_1^2-\varphi_2^2=1.$
3. $u=\varphi(\omega_1,\omega_2),\ \omega_1=x_0;\ \omega_2=\left(x_1^2+x_2^2\right)^{1/2},\ \varphi_1^2-\varphi_2^2=1.$
4. $u=\varphi(\omega_1,\omega_2),\ \omega_1=x_0;\ \omega_2=\left(x_1^2+x_2^2+x_3^2\right)^{1/2},\ \varphi_1^2-\varphi_2^2=1.$
5. $u=\varphi(\omega_1,\omega_2),\ \omega_1=x_0;\ \omega_2=x_3,\ \varphi_1^2-\varphi_2^2=1.$
6. $u^2=-\varphi(\omega_1,\omega_2)+x_0^2,\ \omega_1=\left(x_1^2+x_2^2\right)^{1/2},\ \omega_2=x_3,\ \varphi_1^2+\varphi_2^2-4\varphi=0.$
7. $u^2=\varphi(\omega_1,\omega_2)+x_0^2-x_3^2,\ \omega_1=x_1,\ \omega_2=x_2,\ \varphi_1^2+\varphi_2^2+4\varphi=0.$
8. $u^2=\varphi(\omega_1,\omega_2)-x_3^2,\ \omega_1=x_0,\ \omega_2=\left(x_1^2+x_2^2\right)^{1/2},\ \varphi_1^2-\varphi_2^2-4\varphi=0.$

The ansatzes (1)–(8) can be written in the form (2), where $\omega=(\omega_1(x),\omega_2(x))$ are two-dimensional invariants of the splitting subgroups of the group $\mathrm{P}(1,4)$.

9. $2x_0\omega_1=-\varphi(\omega_1,\omega_2)+x_3^2,\ \omega_1=x_0+u,\ \omega_2=x_2;\ 4\omega_1\varphi_1-\varphi_2^2+4(\omega_1^2-\varphi)=0.$
10. $2x_0\omega_1=-\varphi(\omega_1,\omega_2)+x_1^2+x_2^2,\ \omega_1=x_0+u,\ \omega_2=x_3;\ 4\omega_1\varphi_1-\varphi_2^2+4(\omega_1^2-\varphi)=0.$
11. $2x_0\omega_1=-\varphi(\omega_1,\omega_2)+x_3^2,\ \omega_1=x_0+u,\ \omega_2=\left(x_1^2+x_2^2\right)^{1/2};\ 4\omega_1\varphi_1-\varphi_2^2+4(\omega_1^2-\varphi)=0.$
12. $\arcsin\dfrac{x_2}{\omega_2}+\dfrac{x_3}{\varepsilon\omega_1}=\varphi(\omega_1,\omega_2),\ \omega_1=x_0+u,\ \omega_2=\left(x_1^2+x_2^2\right)^{1/2};\ \varphi_2^2+\omega_2^{-2}+\omega_1^2=0.$
13. $\dfrac{1}{2}\arcsin\dfrac{x_3}{\omega_2}+\dfrac{1}{e}\arcsin\dfrac{x_2}{\omega_1}=\varphi(\omega_1,\omega_2),\ \omega_1=\left(x_1^2+x_2^2\right)^{1/2};\ \omega_2=\left(x_3^2+u^2\right)^{1/2};$
    $\varphi_1^2+\varphi_2^2+e^{-2}\omega_1^{-2}+\frac{1}{4}\omega_2^{-2}=0,\quad e\neq0.$
14. $\arcsin\dfrac{x_2}{\omega_1}-\dfrac{1}{e}\,\mathrm{arch}\,\dfrac{x_0}{\omega_2}=\varphi(\omega_1,\omega_2),\ \omega_1=\left(x_1^2+x_2^2\right)^{1/2};\ \omega_2=\left(x_0^2-u^2\right)^{1/2};$
    $\varphi_1^2+\varphi_2^2+\omega_1^{-2}+e^{-2}\omega_2^{-2}=0,\quad e\neq0.$

The ansatzes (9)–(14) can be written in the form (3), where $\omega=(\omega_1(x,u),\omega_2(x,u))$ are two-dimensional invariants of the splitting subgroups of the group $\mathrm{P}(1,4)$.

15. $\arcsin\dfrac{x_2}{\omega_1} - \dfrac{1}{d}\ln(x_0+u) = \varphi(\omega_1,\omega_2)$, $\omega_1 = \left(x_1^2+x_2^2\right)^{1/2}$; $\omega_2 = \left(u^2+x_3^2-x_0^2\right)^{1/2}$;
$\varphi_1^2+\varphi_2^2+\omega_1^{-2}+2d^{-1}\omega_2^{-1}\varphi_2 = 0, d \neq 0.$

16. $\ln(x_0^2-u^2) + 2e\varepsilon\arcsin\dfrac{x_2}{\omega_2} - 2\varepsilon\,\mathrm{arch}\,\dfrac{x_0}{\sqrt{x_0^2-u^2}} = \varphi(\omega_1,\omega_2)$, $\omega_1 = x_3$,

$\omega_2 = \left(x_1^2+x_2^2\right)^{1/2}$,
$e>0$, $\varepsilon=\pm1$; $\varphi_1^2+\varphi_2^2+4e^2\omega_2^{-2}=0.$

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