Nonlinear Mathematical Physics 1996, V.3, N 3–4, 458–462.

Fundamental Solutions of the Axial Symmetric Goursat Problem

A.A. BORGHARDT, D.Ya. KARPENKO and N.Yu. NOSENKO

Donetsk Physico-Technical Institute of the National Ukrainian Academy of Sciences

Abstract

Fundamental solutions (FS) with a given boundary condition on the characteristics of relativistic problems with axial symmetry are considered. This is so-called the Goursat problem (GP) or zero plane formalism in Dirac's terminology, or modification of the proper time method in the Fock-Nambu-Schwinger formalism (FNS).

Closed FS for the Volkov problem from the point of view of GP can be found. This means that integration over proper time in a FNS integral transformation can be performed. Using the special chosen dynamic symmetry of the initial state, FS for a particle in constant magnetic or constant electric field may also be calculated.

Introduction

In mathematical physics the GP is formulated for a 1-D hyperbolic equation if boundary conditions on the characteristics $\xi = ct - z = \text{const}$, $\eta = ct + z = \text{const}$ are given [1–3]. The FS of GP are called Riemann functions [1, 2]. For example, for the 1-D wave equation (WE) with a positive parameter a^2

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - a^2\right)\psi(z,t) = \left(4\frac{\partial^2}{\partial\xi\partial\eta} + a^2\right)\psi = 0 \tag{1}$$

the Riemann function is [1]

$$G_R = J_0(\sqrt{(c^2t^2 - z^2)a^2}) = J_0(\sqrt{\xi\eta a^2}) \tag{2}$$

where J_0 is the Bessel function of the first kind of order zero. If the parameter a^2 is an elliptic n-D operator then WE (1) is transformed into $(n+1)$-D WE. Let

$$a^2 = k_0^2 - \Delta_\perp \tag{3}$$

where $k_0 = \dfrac{mc}{\hbar}$ is the inverse Compton length of a relativistic particle, and $\Delta_\perp = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is the 2-D Laplace operator then WE (1) becomes the Klein-Fock equation with the corresponding Riemann function [4]

$$G_R = J_0\left(\sqrt{\xi\eta(k_0^2 - \Delta_\perp)}\right) \mid i > \tag{4}$$

where $\mid i >$ is a boundary value of the Riemann function on the characteristics $\xi = 0$, $\eta = 0$. Let the ket-vector $\mid i >$ be the 2-D delta-function

$$\mid i >= \delta(x_1, x_2) = \frac{1}{4\pi^2} \int \exp\{i(k_1 x_1 + k_2 x_2)\} dk_1 dk_2 = \frac{1}{2\pi} \int\limits_0^\infty J_0(kx_\perp) k dk \quad (5)$$

where $x_\perp = \sqrt{x_1^2 + x_2^2}$ and the value $\xi\eta = c^2t^2 - z^2 \geq 0$, which is a time-like interval of the Minkowski space M_+^2, then substitution of Eq.(5) into Eq.(4) gives

$$G_R = G_R^{(+)} = \frac{1}{4\pi^2} \int \exp\{i(k_1 x_1 + k_2 x_2)\} J_0\left(\sqrt{\xi\eta(k_0^2 + k_\perp^2)}\right) dk_1 dk_2 =$$

$$\frac{1}{2\pi}\left(\frac{\delta(\sqrt{\xi\eta} - x_\perp)}{x_\perp} - k_0 \Theta(\sqrt{\xi\eta} - x_\perp)\frac{J_1(k_0\lambda)}{\lambda}\right) \quad (6)$$

where the notations are introduced:

$$k_\perp = \sqrt{k_1^2 + k_2^2}, \quad (7)$$

$\Theta(z)$ is the Heaviside function, and $\lambda = \sqrt{c^2t^2 - z^2 - x_\perp^2} = \sqrt{c^2t^2 - r^2} = \sqrt{\xi\eta - x_\perp^2}$ is a time-like interval of the Minkowski space M_+^4. Taking into account properties of the delta function

$$\frac{\delta(\sqrt{\xi\eta} - x_\perp)}{x_\perp} = \frac{\delta(\mid t \mid - r/c)}{cr} = 2\delta(\lambda^2) \quad (9)$$

and

$$\Theta(\sqrt{\xi\eta} - x_\perp) = \Theta(\mid t \mid - r/c) = \Theta(\lambda^2), \quad (10)$$

we obtain a relativistically invariant representation of the Riemann function $G_R^{(+)}$

$$G_R^{(+)} = \frac{1}{\pi}\left(\delta(\lambda^2) - \frac{k_0}{2}\Theta(\lambda^2)\frac{J_1(k_0\lambda)}{\lambda}\right). \quad (11)$$

The problem, which does not take place for the 1-D WE, arises for the n-D WE (where $n > 1$). We rewrite the formula (4) in the form [4]

$$G_R = J_0(\sqrt{-\xi\eta(\Delta_\perp - k_0^2)}) \mid i > \quad (12)$$

where $-\xi\eta = z^2 - c^2t^2 \geq 0$ is a space-like interval of the Minkowski space M_-^2. It should be noted that the ket-vector $\mid i >$ in (12) cannot be a localized delta function because in this case the operator $J_0 \to I_0$ (where I_0 is a modified Bessel function which increases exponentially). For this reason the ket-vector $\mid i >$ in Eq.(12) has the form

$$\mid i >= \frac{1}{2\pi} \int\limits_{k_0}^\infty K_0(kx_\perp) k dk = k_0 \frac{K_1(k_0 x_\perp)}{2\pi x_\perp} \quad (13)$$

where K_0, K_1 are the MacDonald functions which are eigenfunctions of an elliptic operator $\Delta_\perp$

$$\Delta_\perp K_0(kx_\perp) = k^2 K_0(kx_\perp). \quad (14)$$

Substituting (13) into Eq.(12), we obtain a Riemann function in the space-like $M_-^2 \subset M_-^4$

$$G_R = G_R^{(-)} = \frac{1}{2\pi}\int\limits_{k_0}^{\infty} J_0\left(\sqrt{-\xi\eta(k^2-k_0^2)}\right) K_0(kx_\perp)kdk = \frac{k_0 K_1(k_0\widetilde{\lambda})}{2\pi\widetilde{\lambda}} \qquad (15)$$

where $\widetilde{\lambda} = \sqrt{-\xi\eta + x_\perp^2} = \sqrt{r^2 - c^2t^2}$ is a space-like interval of the space M_-^4.

The linear combination of Riemann functions $G_R^{(\pm)}$ coincides with the causal function Δ^c, but the latter contains the Bessel function of the second kind $N_1(k_0\lambda)$ which is needed to satisfy the radiation condition.

The aim of our paper is to construct Riemann functions for the Klein-Fock equation with electromagnetic interaction preserving the axial symmetry.

1 Riemann function for the Volkov problem

Let us consider the Klein-Fock equation which describes the behaviour of a relativistic particle in a plane electromagnetic wave with the potential

$$A_\mu = A_\mu^\perp = (A_1, A_2, 0, 0) \qquad (1.1)$$

where $A_i = A_i(\xi) = A_i(ct-z)$ $(i = 1,2)$. This is the so-called Volkov problem. A Riemann function of the Volkov problem is

$$G_R^{(+)} = \int e^{ik_ix_i} J_0\left(\sqrt{\xi\eta(k_0^2 + k_i^2 + \frac{2e}{\hbar c} < A_i > k_i + \frac{e^2}{\hbar^2c^2} < A_i^2 >}\right)\frac{dk_1dk_2}{4\pi^2} \qquad (1.2)$$

where $< A_i >= \frac{1}{\xi}\int\limits_0^\xi A_i(z)dz$. To perform the integration in Eq.(1.2), we shall make a shift of the momentum

$$k_i + \frac{e}{\hbar c} < A_i >= q_i \qquad (1.3)$$

which eliminates the linear electromagnetic term, and Eq.(1.2) has the form (see the relation (6))

$$G_R^{(+)} = \exp\left\{-\frac{ie}{\hbar c} < A_i > x_i\right\}\int\limits_0^\infty J_0(qx_\perp)J_0(\sqrt{\xi\eta(q^2+k^2(\xi))}\,\frac{qdq}{2\pi} =$$

$$\frac{1}{2\pi}\exp\{\frac{-ie}{\hbar c} < A_i > x_i\}\left(\frac{\delta(\sqrt{\xi\eta}-x_\perp)}{x_\perp} - \frac{k(\xi)\Theta(\sqrt{\xi\eta}-x_\perp)J_1(k(\xi)\lambda)}{\lambda}\right) \qquad (1.4)$$

where $k(\xi) = k_0\sqrt{1 + \frac{e^2}{\hbar^2c^2}\,(< A_i^2 > - < A >^2)}$. From the FS of the Volkov problem it follows that in the presence of the wave the proper mass of the particle increases.

2 Riemann functions for a particle in constant uniform magnetic or electric field

The constant and uniform magnetic field H may be described by the vector potential

$$A_\mu^\perp = (A_1, A_2, 0, 0), \quad A_1 = -\frac{Hx_2}{2}, \quad A_2 = \frac{Hx_1}{2} \tag{2.1}$$

and the Riemann function of GP is

$$G_R^{(+)} = J_0\left(\sqrt{\xi\eta(k_0^2 - D_\mu^{T^2})}\right) \mid i >, \tag{2.2}$$

where

$$D_\mu^T = \partial_\mu^T + \frac{ie}{\hbar c} A_\mu^T. \tag{2.3}$$

For the sake of simplicity we take the boundary condition for $\mid i >$ in the form

$$\mid i >= \sqrt{\frac{\hbar c}{eH}} \ \exp\left\{-\frac{\alpha x_\perp^2}{2}\right\} L_n(\alpha x_\perp^2) \tag{2.4}$$

where $\alpha = \dfrac{eH}{2\hbar c}$, and $L_n(z)$ is the Laguerre polynomial.

Substituting Eq.(2.5) into Eq.(2.2), we obtain

$$G_R^{(+)} = \exp\left\{-\frac{\alpha x_\perp^2}{2}\right\} \sum_n J_0\left(\sqrt{\xi\eta k_0^2\left(1 + \frac{H}{H_0}(2n+1)\right)}\right) L_n(\alpha x_\perp^2) \tag{2.5}$$

where $H_0 = \dfrac{m^2c^3}{e\hbar}$. Using the asymptotics of a Laguerre polynomial with large n

$$\exp\left\{-\frac{\alpha x_\perp^2}{2}\right\} L_n(\alpha x_\perp^2) \simeq J_0(\sqrt{4\alpha n} x_\perp) \tag{2.6}$$

one can carry out approximate summation into (2.5) according to the rule [6]

$$4\alpha n \to q^2, \qquad \sum_n \to \int_0^\infty q dq \tag{2.7}$$

and then (see the relation (6))

$$G_R^{(+)} \simeq \int_0^\infty J_0\left(\sqrt{\xi\eta(q^2 + k^2(H))}\right) J_0(qx_\perp) \ \frac{qdq}{2\pi} =$$

$$\frac{1}{2\pi}\left[\frac{\delta(\sqrt{\xi\eta} - x_\perp)}{x_\perp} - k(H)\Theta\left(\sqrt{\xi\eta} - x_\perp\right) \ \frac{J_1(k(H)\lambda)}{\lambda}\right] \tag{2.8}$$

where $k(H) = k_0\sqrt{1 + \frac{H}{H_0}}$. Thus, in a constant magnetic field, the proper mass of the particle increases by $\sqrt{1 + \frac{H}{H_0}}$ times.

Next we consider the Riemann function for a moving particle in the constant electric field E defined by the vector potential

$$A_\mu = (0, 0, A_3, iA_0), \quad A_3 = -\frac{ctE}{2}, \quad A_0 = -\frac{Ez}{2}. \tag{2.9}$$

For a constant electric field there are two types of solutions which differ by a charge sign, and the Riemann function may be constructed from these partial solutions [4]

$$G_R = G_R^{(-)} = \sum_n \exp\left\{\pm\frac{ieE(z^2 - c^2t^2)}{4\hbar c}\right\} L_n\left(\mp\frac{ieE}{2\hbar c}(z^2 - c^2t^2)\right) K_0(k_n x_\perp), \tag{2.10}$$

where

$$k_n = k_0\sqrt{1 \pm \frac{iE}{E_0}(2n+1)}, \quad E_0 = H_0. \tag{2.11}$$

If we use the Laguerre polynomial asymptotics for large n again, then approximate summation may also be performed in (2.11) (see the relation (15))

$$G_R^{(-)} \simeq \int\limits_{k(e)}^{\infty} J_0\left(\sqrt{-\xi\eta(k_\perp^2 - k^2(E))}\right) K_0(k_\perp x_\perp)\,\frac{k_\perp dk_\perp}{2\pi} = k(E)\,\frac{K_1(k(E)\widetilde{\lambda})}{2\pi\widetilde{\lambda}}, \tag{2.13}$$

where

$$k(E) = k_0\sqrt{1 \pm \frac{iE}{E_0}} = \frac{mc}{\hbar}\sqrt{1 \pm \frac{iE}{E_0}}. \tag{2.12}$$

In a constant electric field the stable Riemann function exists in the space-like $M_-^2 \subset M_-^4$ and the proper mass of the particle gets a small imaginary addition. It is possible, because for the constant electric field as distinct from the field of a plane electromagnetic wave and constant magnetic field has the ability to create pairs.

In conclusion we note that the new classification of the FS of WE from the point of view of the GP makes possible investigating solution singularities. Namely, if the boundary state $\mid i >$ is localized, then a Riemann function exists inside the light sector (cone). And if the boundary state $\mid i >$ is nonlocalized, then the Riemann function exists outside the light sector (cone).

References

[1] Hadamard J., The Cauchy Problem for Linear Equations with Partial Derivatives, Paris, 1938.

[2] Courant R., Partial Differential Equations, New York, London, 1962.

[3] Hillion P., *J. Math. Phys.,* 1990, V.31, N 8, 1939–1942.

[4] Borghardt A.A., Karpenko D.Ya., *Differentsialnye Uravneniya,* 1984, V.20, N 2, 302–308 (in Russian).

[5] Borghardt A.A., Karpenko D.Ya., *Dopovidi Ukrain. Acad. Nauk,* 1992, N 2, 19–22.

[6] Borghardt A.A., Karpenko D.Ya., *Ukrain. Fiz. Zhurn.,* 1987, V.32, 1324–1334 (in Russian).