Nonlinear Mathematical Physics 1996, V.3, N 3–4, 417–420.

On Exact Solutions of the Lorentz-Maxwell Equations

Igor REVENKO

Institute of Mathematics of the National Ukrainian Academy of Sciences, 3 Tereshchenkivs'ka Street, Kyiv 4, Ukraina

Abstract

New exact solutions are obtained for the systems of classical electrodynamics equations.

Motion of a classical spinless particle moving in electromagnetic field is described by the system of ordinary differential equations (Lorentz) and of partial differential equations (Maxwell) [1]

$$mu_\mu = eF_{\mu\nu}u^\nu, \qquad u_\mu \equiv \dot{x}_\mu = \frac{dx_\mu}{d\tau}, \tag{1}$$

where $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x^\mu} - \dfrac{\partial A_\mu}{\partial x^\nu}$ is the tensor of electromagnetic field

$$\partial_\nu \partial^\nu A_\mu - \partial^\mu (\partial_\nu A_\nu) = j_\mu, \qquad j_\mu = eu_\mu, \tag{2}$$

$$u_\mu u^\mu = 1, \tag{3}$$

A_μ is the potential of electromagnetic field. Some exact solutions of system (1) ,(2) are found in [2].

In the present paper using symmetry properties of (1),(2), we have obtained new classes of exact solutions of the Lorentz-Maxwell system.

1. We choose the electromagnetic potential A_μ as follows:

$$\begin{aligned} &A_0 = \rho(\omega)\theta + \sigma(\omega)\theta^{-1}, \quad A_1 = A_1(\omega), \quad A_2 = A_2(\omega), \\ &A_3 = \rho(\omega)\theta - \sigma(\omega)\theta^{-1}, \quad \theta = x_0 + x_3, \quad \omega = x_1 - \alpha \ln|\theta|, \end{aligned} \tag{4}$$

where ρ, σ, A_1, A_2 are arbitrary smooth enough functions depending on the variable ω only. The Lagrangian L of equation (1)

$$L = \frac{m}{2}\dot{x}_\mu \dot{x}^\mu + e\dot{x}^\mu A_\mu \tag{5}$$

for the field (4) is invariant under the three-dimensional Lie algebra having the basis elements

$$< x_0\partial_3 + x_3\partial_0 + \alpha\partial_1, \partial_0 - \partial_3, \partial_2 > . \tag{6}$$

It follows from the Noether theorem that the functions

$$mu_3 + eA_3 + mu_0 + eA_0 = C_1, \qquad -mu_2 - eA_2 = C_2,$$
$$x_0\,(-mu_3 - eA_3) + x_3\,(mu_0 + A_0) + \alpha\,(-mu_1 - eA_1) = C_3, \tag{7}$$

where C_1, C_2, C_3 are arbitrary constants, are integrals of motion of equation (1) for the field (4).

Equations (2) for the field (4) are of the form

$$eu_0 = -\rho''\theta + \theta^{-1}\left\{-\sigma'' + \alpha(2\rho' - 2\alpha\rho'' + A_1'')\right\},$$
$$eu_1 = 2(\rho' - \alpha\rho''), \qquad eu_2 = -A_2'', \tag{8}$$
$$eu_3 = -\rho''\theta + \theta^{-1}\left\{-\sigma'' + \alpha(2\rho' - 2\alpha\rho'' + A_1'')\right\}.$$

Using the motion integral (7), we rewrite system (8) as follows:

$$A_2'' = \frac{m}{e} - eA_2 = C_2, \quad \rho''\frac{m}{e} - e\rho = 0, \quad C_1 = 0,$$
$$(\alpha A_1 - \sigma)''\frac{m}{e} - e(\alpha A_1 - \sigma) = C_3. \tag{9}$$

By direct verification one can become convinced of the fact that the functions

$$\alpha A_1 - \sigma = a_0 \exp\left\{\frac{e}{\sqrt{m}}\omega\right\} + b_0 \exp\left\{-\frac{e}{\sqrt{m}}\omega\right\} - \frac{C_3}{e},$$
$$\rho = a_1 \exp\left\{\frac{e}{\sqrt{m}}\omega\right\} + b_1 \exp\left\{-\frac{e}{\sqrt{m}}\omega\right\}, \tag{10}$$
$$A_2 = a_2 \exp\left\{\frac{e}{\sqrt{m}}\omega\right\} + b_2 \exp\left\{-\frac{e}{\sqrt{m}}\omega\right\} - \frac{C_2}{e}$$

satisfy system (9).

As the vector u_μ satisfy relation (3) we should impose an additional condition on the functions ρ, σ, A_1, A_2

$$4\rho''(\sigma'' - \alpha A_1'') + 4\rho''^2\alpha^2 - 4\rho'^2 - A_2''^2 = e^2. \tag{11}$$

Substituting expressions (10) into (11), we get the relations for constants a_i, b_i

$$4a_1a_0 + 4(\alpha^2 - m)a_1^2 - a_2^2 = 0, \quad a_1b_1 \neq 0,$$
$$4b_1b_0 + 4(\alpha^2 - m)b_1^2 - b_2^2 = 0, \tag{12}$$
$$4a_1b_0 + 4b_1a_0 + 8a_1b_1(\alpha^2 + m) - 2b_2a_2 = \frac{m^2}{e^2}.$$

To construct solutions of equation (1), (4), we make the change of variables

$$y_0 = x_0 + x_3, \quad y_1 = x_1 - \alpha \ln|x_0 + x_3|, \quad y_2 = x_2, \quad y_3 = x_0 - x_3. \tag{13}$$

Then the motion equations take the form

$$\frac{dy_0}{d\tau} = u_0 + u_3 = -\frac{2\rho'' y_0}{e}, \quad \frac{dy_1}{d\tau} = \frac{2\rho'}{e}, \quad \frac{dy_2}{d\tau} = u_2 = -\frac{A_2''}{e},$$
$$\frac{dy_3}{d\tau} = \frac{2}{ey_0}\left\{-\sigma'' + \alpha A_1'' + \alpha(2\rho' - 2\alpha\rho'')\right\}. \tag{14}$$

Solutions of (14) are given by the quadratures

$$\int \frac{dy_1}{2\rho'} = \frac{\tau}{e} + C_0, \qquad y_0 = \frac{C_4}{\rho'}, \quad y_2 = -\int \frac{A_2'' dy_1}{2\rho'} + C_6,$$
$$y_3 = \frac{1}{C_4}\left\{-\sigma' + \alpha A_1' + \alpha(2\rho - 2\alpha\rho')\right\} + C_5, \tag{15}$$

where C_0, C_4, C_5, C_6 are integration constants.

Thus, exact solutions of system (1), (2) are given by formulae (4), (10), (12), (15).

2. To construct another class of exact solutions of equations (1), (2), we choose the electromagnetic potential as follows:

$$A_0 = \sigma(\omega)\theta + \theta^{-1}\left\{\psi(\omega)\theta_1 + \sigma(\omega)\theta_1^2 + \varphi(\omega)\right\},$$
$$A_1 = 2\sigma(\omega)\theta_1 + \psi(\omega), \quad A_2 = A_2(\omega),$$
$$A_3 = \sigma(\omega)\theta + \theta^{-1}\left\{\psi(\omega)\theta_1 + \sigma(\omega)\theta_1^2 - \varphi(\omega)\right\},$$
$$\theta = x_0 + x_3, \quad \theta_1 = x_1 - \beta\ln|\theta|, \quad \omega = x_2 - \alpha\ln|\theta|, \tag{16}$$

where $\sigma, \psi, \varphi, A_2$ are arbitrary functions on ω.

With such a choice of the electromagnetic potential, Lagrangian (5) is invariant with respect to the algebra

$$< (x_0 + x_3)\partial_1 + x_1(\partial_0 - \partial_3), \quad x_0\partial_3 + x_3\partial_0 + \beta\partial_1 + \alpha\partial_2, \quad \partial_0 - \partial_3 >$$

and, consequently, equations (1) admit three integrals of motion

$$(x_0 + x_3)(-mu_1 - eA_1) + x_1(mu_0 + eA_0 + mu_3 + eA_3) = C_1,$$
$$mu_0 + eA_0 + mu_3 + eA_3 = C_3,$$
$$x_0(-mu_3 - eA_3) + x_3(mu_0 + eA_0) + \beta(-mu_1 - eA_1) +$$
$$\alpha(-mu_2 - eA_2) = C_2. \tag{17}$$

Substituting (16) into (2), we find the four–vector u_i

$$eu_0 = -\sigma''\theta + \theta^{-1}\left\{\psi''\theta_1 + [-\varphi'' - 2\sigma + \alpha(A_2'' + 4\sigma' - 2\alpha\sigma'')] - \theta_1^2\sigma''\right\},$$
$$eu_1 = -2\sigma''\theta_1 - \psi'', \quad eu_2 = 4\sigma' - 2\alpha\sigma'', \tag{18}$$
$$eu_3 = -\sigma''\theta - \theta^{-1}\left\{-\psi''\theta_1 + [-\varphi'' - 2\sigma + \alpha(A_2'' + 4\sigma' - 2\alpha\sigma'')] - \theta_1^2\sigma''\right\}.$$

Normalizing 4-vector u_μ according to (3), we arrive at the following condition for the functions $\sigma, \psi, \varphi, A_2$:

$$4\sigma''(\varphi'' - \alpha A_2'') + 8\sigma''\sigma + 8\alpha^2\sigma''^2 - \psi''^2 - 16\sigma'^2 = e^2. \tag{19}$$

Compatibility of equations (17), (18) is ensured by the following conditions:

$$\sigma'' - \frac{e^2}{m}\sigma = 0, \quad \psi'' - \frac{e^2}{m}\psi = 0, \quad C_1 = 0, \quad C_2 = 0,$$
$$(\varphi - \alpha A_2)'' - \frac{e^2}{m}(\varphi - \alpha A_2) = C_3\frac{e}{m}. \tag{20}$$

General solutions of equations (20) read

$$\begin{aligned}
\sigma &= a_0 \exp\left\{\frac{e}{\sqrt{m}}\omega\right\} + b_0 \exp\left\{-\frac{e}{\sqrt{m}}\omega\right\},\\
\psi &= a_1 \exp\left\{\frac{e}{\sqrt{m}}\omega\right\} + b_1 \exp\left\{-\frac{e}{\sqrt{m}}\omega\right\},\\
\varphi - \alpha A_2 &= a_2 \exp\left\{\frac{e}{\sqrt{m}}\omega\right\} + b_2 \exp\left\{-\frac{e}{\sqrt{m}}\omega\right\} - \frac{C_3}{e}
\end{aligned} \tag{21}$$

where a_i, b_i are arbitrary constants.

To satisfy equation (19), constants a_i, b_i have to obey the conditions

$$\begin{aligned}
&4\left(a_0 a_2 + 2a_0^2\left(\alpha^2 - \frac{m}{e^2}\right)\right) - a_1^2 = 0,\\
&4\left(b_0 b_2 + 2b_0^2\left(\alpha^2 - \frac{m}{e^2}\right)\right) - b_1^2 = 0,\\
&4\frac{e^2}{m^2}(a_0 b_2 + b_0 a_2) + 16 a_0 b_0\left(\frac{3}{m} + \frac{e^2\alpha^2}{m^2}\right) - 2\frac{e^2}{m^2}a_1 b_1 = 1.
\end{aligned} \tag{22}$$

In the curvilinear coordinate system

$$y_0 = x_0 + x_3,\ \ y_1 = \frac{x_1}{x_0 + x_3},\ \ y_2 = x_2 - \alpha \ln|x_0 + x_3|,\ \ y_3 = x_0 - x_3$$

equations of motion of a particle take a form

$$\begin{aligned}
&\frac{dy_0}{d\tau} = -\frac{2\sigma'' y_0}{e},\quad \frac{dy_1}{d\tau} = \frac{2\sigma''\beta\ln|y_0| - \psi''}{e y_0},\quad \frac{dy_2}{d\tau} = \frac{4\sigma'}{e},\\
&\frac{dy_3}{d\tau} = \frac{2}{y_0}\left\{-\psi''\left[y_1 y_0 - \beta\ln|y_0|\right] + \right.\\
&\qquad \left.(-\varphi - 2\sigma + \alpha(A_2'' + 4\sigma' - 2\alpha\sigma'')) - (y_1 y_0 - \beta\ln|y_0|)^2\sigma''\right\}.
\end{aligned} \tag{23}$$

Solutions of equations (23) are given by quadratures

$$\begin{aligned}
&\int \frac{dy_2}{4\sigma'} = \frac{\tau}{e} + C_0,\quad y_0 = C_4(\sigma')^{-1/2},\\
&y_1 = \int \frac{\left\{2\sigma''\beta\ln\left[C_4(\sigma')^{-1/2}\right] - \psi''\right\} dy_2}{4C_4(\sigma')^{-1/2}} + C_5 = K(y_2) + C_5,\\
&y_3 = \int\left\{-\psi''\left[(K + C_5)C_4(\sigma')^{-1/2} - \beta\ln|C_4(\sigma')^{-1/2}|\right] + \right.\\
&\qquad (-\varphi'' - 2\sigma + \alpha(A_2'' + 4\sigma' - 2\alpha\sigma'')) -\\
&\qquad \left.\sigma''\left[(K + C_5)C_4(\sigma')^{-1/2} - \beta\ln|C_4(\sigma')^{-1/2}|\right]^2\right\}\frac{dy_2}{2C_4(\sigma')^{-1/2}} + C_6.
\end{aligned}$$

Thus, exact solutions of equations (1), (2) are given by formulae (16), (21), (22), (24).

References

[1] Meller K., Relativity Theory, Moscow, Atomizdat, 1975, 400p.

[2] Bagrov V.G., Gitman D.M., Ternov I.M., Shapovalov V.N., Exact Solutions of Relativistic Wave Equations, Novosibirsk, Nauka, 1982, 144p.