Alternative Approaches to the Modelling of Volatility in European Option Valuation

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***Abstract*—This chapter uses four different models to value the option price: Random Walk (RW), ARCH, GARCH and TARCH. Each model is applied within a Monte-Carlo framework. I attempt to identify the best model in terms of their ability to predict the market price of the option**

Keywords-euyropean option; random walk; arch; garch; tarch; monte carlo simulation

I. INTRODUCTION

Option pricing theory has a long and illustrious history. In 1973, Fischer Black and Myron Scholes presented the first satisfactory equilibrium option price model. The formula they developed has come to be known as the "Black-Scholes formula" and to this day remains an industry standard for option valuation. Robert Merton, among many others, extended the Black-Scholes model in several important ways. As these studies have shown, option pricing theory is relevant to almost every area of finance. The principal attraction of options and other derivative securities is their potential to limit the risk associated with unexpected falls in the price of the underlying asset. By combining an option with the underlying asset in the optimal way, it is possible, in theory at least, to create the "perfect hedge".

Two basic types of option on the stock market are European and American options. The fundamental difference is that European options can only be exercised at expiry, while American options may be exercised at any date on or before expiry (i.e. "early exercise" is permitted). Ceteris paribus, an American option has a higher value than a European option, as a consequence of the "privelige" of early exercise. The problem of valuing European options is much easier than for American options. The basic Black-Scholes formula is a closed-form formula that can be applied directly to the problem of valuing a European option. Valuation of American options requires more sophisticated approaches including simulation.

In this chapter, there is a much greater focus on the valuation of European options. However, the framework developed in this chapter (particularly the Monte-Carlo routines) can in principle be extended to the case of American options, and this is the plan for the next chapter.

II. LITERATURE REVIEW

Volatility is the only one that as unknown and therefore somehow needs to be estimated or modeled. Finance researchers have devoted large amounts of effort to this problem. Basically there are two approaches. One is to work with the deterministic volatility function (DVF); the other assumes stochastic volatility. Bothe approaches are based on the Black-Scholes assumption.

Many models commence by assuming that the underlying stock price S follows the following process, $ds = \mu S dt + \sigma S dz$

Where dz is a Wiener process, σ is constant volatility. Given this assumption, it can be shown that the values of a European call option (c_t) is given by the Black-Scholes equation (Hull, 2005):

$$c_t = S_t \Phi(d_1) - \exp(-r\tau) K \Phi(d_2)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Conversely, given a time-t, stock price S and option price c (K, T), we can define the implied volatility: σ, as the value of which makes the Black-Sholes value equal to the market price.

If this model is correct, and if the option market is perfectly functioning, all options on the same underlying asset with same expiry date but different strike price should, logically, have the same implied volatility (Rubinstein, 1994). In reality, however, this is not the case. Rubinstein (1994) used S&P 500 index from 1986 to 1992 to test the Black-Scholes model. He found that the Black-Scholes model worked quite well during 1986. After that period, a "sneer" (Webster, 1994) appears: options deep in-the-money has higher implied volatility (and is

therefore over-priced) than at-the-money options. Besides Rubinstein, other researchers also find similar evidence in other option markets. .

Because of the limitation of Black-Scholes model exposed by these patterns in implied volatility, researchers have set out to find superior models. These superior models amount to superior assumptions about the volatility of the underlying. A popular one is the deterministic volatility function (DVF). The basic idea of DVF approach is that the volatility is a deterministic function of asset price and time. The model assumes that the market prices of options are given, and then generate implied binomial or trinomial trees consistent with the option prices (Michael, 2002).

However, the more financial researchers worked on volatility, the more it became evident that the deterministic volatility function was not correct, mainly due to the persistence of "smile" or "skew" patterns seen when implied volatilities were computed. Given this, researchers began to use stochastic volatility in place of deterministing volatility (Frey, 1997). The stochastic volatility model (SV) assumes that the volatility is only imperfectly correlated with the asset price process (Frey, 1997). The first SV model was introduced by Hull and White (1987). It assumes that the square of the volatility follows a geometric Brownian motion. Wiggins (1987) assumes that logarithm of σ follows an AR (1) process with a homoscedastic error term. Stein and Stein (1991) also worked with stochastic volatility. Their model has the same assumption as the Wiggins model, the volatility followed an AR (1) process.

Heston (1993) found the limitations of Wiggins and Stein's model. They both assume that the volatility is uncorrelated with spot returns; it can not capture skewness effects of the option market (Heston, 1993).

Another popular stochastic volatility model was autoregressive conditional heteroscedasticity (ARCH) model, which was proposed by Engle (1982). It assumes a linear dependence of conditional variance and on squared past residuals, Bollerslev (1986) proposed the generalized autoregressive conditional heteroscedasticity (GARCH) model based on the ARCH model. It said that the conditional variance of the error term is a linear function of the lagged squared residuals and the lagged residual conditional variance.

More asymmetric volatility models have been proposed by many researchers, for example, EGARCH model—exponential GARCH was introduced by Nelson(1991),The TARCH model ("threshold ARCH") was introduced by Zakoian (1993) and Glosten Jaganathan and Runkle (1993). It deals with asymmetry, by assuming that volatility is boosted more by downside shocks than by upside shocks.

The methods used to price the option can be divided into analytical and numerical methods. Given certain assumptions, analytical methods are applicable to the valuation of European options. However, when these assumptions do not hold, for example, under some forms of stochastic volatility, analytical solutions are not available, and numerical methods must be used instead. For American options, it is nearly always the case that numerical options are required.

The most important analytical method for valuing European options (both call and put) was developed by Fischer Black and Merton Scholes (1973), who were in 1997 awarded the Nobel Prize in Economic Sciences, for this contribution. The simplest form of the Black-Scholes formula for the value of a European call (ct) and a European put (pt) is (Hull, 2005).

Motivated by this volatility consideration, after Black-Scholes, Heston (1993) derived a closed-form solution for the price of a European call option on an asset with stochastic volatility. In his paper, he pointed that Stein (1991) has also generated a new model called the stochastic volatility model to calculate the value of European options. Thanks to contributions to the stochastic volatility function, the second method, ARCH and GARCH model, to calculate the value of options was introduced. Duan (1995) proposed the GARCH model for the option price. The basic idea of his paper was that, firstly, it assumed that the stock price followed GARCH model, then using the GARCH model to find the stock price at expiry data, finally, using Monte Carlo simulation to calculate the price model. After Duan, Heston and Nandi (1997) also used GARCH model to price the options. Instead of using Monte Carlo simulation, they substitute the GARCH model into Black-Scholes formula, and make a new assumption for Black-Scholes model, saying that the volatility is stochastic instead of constant. Recently, Engle, Barone-Adesi and Mancini (2008) proposed a new method for pricing options based on GARCH model with filtered historical simulation. The model used in this paper was the threshold ARCH (TARCH) model, which was introduced by Zakoian (1993), and Glosten Jaganathan and Runkle (1993).

III. STOCHASTIC VOLATILITY MODELS ESTIMATED USING HISTORICAL RETURN DATA

A. *Estimation of random walk*

The first model that is estimated is the random walk model, defined as follows:

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \varepsilon_t$$

$$V(\varepsilon_t) = \sigma^2$$

Where S_t is the underlying price at time t.

B. *Estimation of ARCH (1)*

Model 2: we assume that volatility follows an ARCH (1) process.

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \varepsilon_t$$

$$\sigma_t^{\ 2} = \alpha_0 + \alpha\varepsilon_{1t-1}^{\ 2}$$

C. *Estimation of GARCH (1, 1)*

Model 3: we assume that volatility follows a GARCH (1, 1) process.

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \varepsilon_t$$

$$\sigma_t^{\ 2} = \alpha_0 + \alpha\varepsilon_{1t-1}^{\ 2} + \beta_1\sigma_{t-1}^{\ 2}$$

The results are shown blow

D. *Estimation of TARCH(1, 1)*

Model 4: we assume that volatility follows a TARCH (1, 1) process:

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \varepsilon_t$$

$$\sigma_t^{\ 2} = \alpha_0 + +\beta_1\sigma_{t-1}^{\ 2} + \alpha_1\varepsilon_{t-1}^{\ 2} + \gamma I_{1t-1}\varepsilon_{t-1}$$

Where I_{t-1} is a dummy variable.

$$I_{t-1} = 1, where \varepsilon_{t-1} > 0$$

$$I_{t-1} = 0, where \varepsilon_{t-1} < 0$$

E. *Summary the regression result*

Table 1: summary of regression result:

	Random walk	ARCH	GARCH	TARCH
σ	0.0284			
α_0		0.00074** (0.00004)	0.0000186* (0.0000107)	0.000027** (0.00000195)
α_1		0.0794** (0.026)	0.071** (0.0208)	0.135** (0.034)
β_1			0.0906** (0.03)	0.910** (0.0289)
γ				−0.155** (0.0364)
Log Likelihood	1609.1547	1612.267	1642.439	1656.246

From the above table, we can conclude that TARCH model will be the best model to represent the volatility process, since all coefficients are strongly significant and it has the highest log-likelihood value. Note also that the estimate of γ in the TARCH model is significantly negative, confirming that downward spikes have a greater impact on the next period's volatility

IV. SIMULATION THE ESTIMATED DATA

A. *Simulation of random walk data*

Here, we simulate data from the random walk model estimated in 5.1:

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \varepsilon_t$$

$$V(\varepsilon_t) = 0.0284^2$$

To simulate such a return series, we simply use:

$$\hat{r}_{RW,t} = 0.0284 * z_t$$
$$where: z_t \in N[0,1]$$

B. *Simulation of ARCH (1) data*

The ARCH (1) data is simulated using

$$\hat{r}_{ARCH,t} = z_t\sqrt{\alpha_0 + \alpha_1\hat{r}_{ARCH,t=1}^{\ 2}}$$
$$where: z_t \in N[0,1]$$

The estimates obtained in 5.2 are used in the simulation. That is:

$$\hat{r}_{ARCH,t} = z_t\sqrt{0.00074 + 0.0794\hat{r}_{ARCH,t=1}^{\ 2}}$$
$$where: z_t \in N[0,1]$$

C. *Simulation of GARCH(1, 1) data*

The GARCH (1, 1) model is simulated using:

$$\sigma_{GARCH,t}^{\ 2} = \alpha_0 + \sigma_{GARCH,t-1}^{\ 2} * \left(\alpha_1 z_{t-1}^{\ 2} + \beta_1\right)$$

$$\hat{r}_{GARCH,t} = z_t * \sqrt{\alpha_0 + \sigma_{GARCH,t-1}^{\ 2} * \left(\alpha_1 z_{t-1}^{\ 2} + \beta_1\right)}$$

The estimates obtained in 5.3 are used in the simulation. That is:

$$\sigma_{GARCH,t}^{\ 2} = 0.000186 + \sigma_{GARCH,t-1}^{\ 2} * \left(0.071 z_{t-1}^{\ 2} + 0.906\right)$$

$$\hat{r}_{GARCH,t} = z_t * \sigma_{GARCH,t}$$

D. Simulation of TARCH (1,1) data

I estimate that the volatility follows a TARCH (1, 1) process:

$$\sigma_{TARCH,t}^{2} = \alpha_0 + \sigma_{TARCH,t-1}^{2} * \left(\alpha_1 z_{t-1}^{2} + \beta_1 + \gamma * (z_{t-1} > 0) * z_{t-1}^{2}\right)$$

$$\hat{r}_{TARCH,t} = z_t * \sigma_{TARCH,t}$$

The estimates obtained in 5.3 are used in the simulation. That is:

$$\sigma_{TARCH,t}^{2} = 0.000027 + \sigma_{TARCH,t-1}^{2} * \left(0.135 z_{t-1}^{2} + 0.910 - 0.155 * (z_{t-1} > 0) * z_{t-1}^{2}\right)$$

$$\hat{r}_{TARCH,t} = z_t * \sigma_{TARCH,t}$$

V. MONTE CARLO SIMULATION

A. Simulation for Valuing European option

Random Walk simulation

For present purposes, assume that the option is traded on day t=0, and the expiry date is T. At expiry T, the underlying price will be:

$$S_{RW,T} = \exp\left(\ln(S_0) + \sum_{t=1}^{T} \ln(1 + r_{RW,t})\right)$$

Where: S_0 is the underlying price at the beginning date; $\sigma_t = sd(r_t)$; $r_{RW,t} = \sigma_t * z$

We assume that the strike price is K, the payoff of a European call option will be:

$$payoff\ f_{call_{RW}} = E(\max[S_{RW,T} - K, 0])$$

Payoff of a European put option will be: $payoff\ f_{put_{RW}} = E(\max[K - S_{RW,T}, 0])$

If you discount the expected payoff by risk-free interest rate, you will get the current value of the European option.

ARCH Model simulation

$$S_{ARCH,T} = \exp\left(\ln(S_0) + \sum_{t=1}^{T} \ln(1 + r_{ARCH,t})\right)$$

Where:

$$\hat{r}_{ARCH,t} = z_t \sqrt{0.00074 + 0.0794 \hat{r}_{ARCH,t=1}^{2}}$$

$$where: z_t \in N[0,1]$$

The payoff of a European call option will be:

$$payoff\ f_{call_{ARCH}} = E(\max[S_{ARCH,T} - K, 0])$$

Payoff of a European put option will be: $payoff\ f_{put_{ARCH}} = E(\max[K - S_{ARCH,T}, 0])$

If you discount the expected payoff by risk-free interest rate, you will get the current value of the European option.

GARCH Model simulation

$$S_{GARCH,T} = \exp\left(\ln(S_0) + \sum_{t=1}^{T} \ln(1 + r_{ARCH,t})\right)$$

$$\sigma_{GARCH,t}^{2} = 0.000186 + \sigma_{GARCH,t-1}^{2} * \left(0.071 z_{t-1}^{2} + 0.906\right)$$

The payoff of a European call option will be:

$$payoff\ f_{call_{GARCH}} = E(\max[S_{GARCH,T} - K, 0])$$

Payoff of a European put option will be: $payoff\ f_{put_{GARCH}} = E(\max[K - S_{GARCH,T}, 0])$

If you discount the expected payoff by risk-free interest rate, you will get the current value of the European option.

TARCH Model simulation

$$S_{TARCH,T} = \exp\left(\ln(S_0) + \sum_{t=1}^{T} \ln(1 + r_{TARCH,t})\right)$$

Where:

$$\sigma_{TARCH,t}^{2} = 0.000027 + \sigma_{TARCH,t-1}^{2} * \left(0.135 z_{t-1}^{2} + 0.910 - 0.155 * (z_{t-1} > 0) * z_{t-1}^{2}\right)$$

The payoff of a European call option will be:

$$payoff\ f_{call_{TARCH}} = E(\max[S_{TARCH,T} - K, 0])$$

Payoff of a European put option will be: $payoff\ f_{put_{TARCH}} = E(\max[K - S_{TARCH,T}, 0])$

If you discount the expected payoff by risk-free interest rate, you will get the current value of the European option.

B. Using four simulations to estimate the European option price

The following table shows the call option values obtained from the Monte Carlo simulation for each model. We see that the value computed using the TARCH model is the one that is closest to the market price.

Table 2: estimated value and market price of call options:

Date	tau	Underlyin	Strike	rf	RW	ARCH	GARCH	TARCH	BS	Call_bid
02-Jan-2002	45	5.2398	5.2	0.024	0.419	0.413	0.26	0.29	0.314	0.291
03-Jan-2002	44	5.4835	5.2	0.02455	0.564	0.561	0.414	0.449	0.463	0.463
04-Jan-2002	43	5.534	5.2	0.024406	0.6	0.594	0.456	0.491	0.495	0.494
07-Jan-2002	40	5.4731	5.2	0.0243	0.539	0.533	0.388	0.423	0.443	0.439
08-Jan-2002	39	5.4911	5.2	0.02345	0.549	0.54	0.397	0.43	0.451	0.448
09-Jan-2002	38	5.4836	5.2	0.023713	0.552	0.544	0.392	0.424	0.443	0.434
10-Jan-2002	37	5.4213	5.2	0.0235	0.409	0.483	0.336	0.369	0.397	0.389
11-Jan-2002	36	5.3504	5.2	0.0234	0.977	0.973	0.9	0.905	0.802	0.336
14-Jan-2002	33	5.2467	5.2	0.0213	0.364	0.356	0.206	0.233	0.275	0.261
15-Jan-2002	32	5.2852	5.2	0.021475	0.497	0.493	0.345	0.378	0.386	0.279
16-Jan-2002	31	5.0901	5.2	0.0213	0.28	0.27	0.12	0.14	0.187	0.17
17-Jan-2002	30	5.2506	5.2	0.022112	0.348	0.34	0.192	0.218	0.265	0.254
18-Jan-2002	29	5.0987	5.2	0.022875	0.27	0.263	0.115	0.135	0.184	0.155
22-Jan-2002	25	4.9133	5.2	0.02315	0.55	0.546	0.111	0.449	0.395	0.077
23-Jan-2002	24	5.0244	5.2	0.023338	0.204	0.197	0.064	0.078	0.132	0.104
24-Jan-2002	23	5.1093	5.2	0.023387	0.245	0.236	0.091	0.109	0.163	0.127
25-Jan-2002	22	5.0875	5.2	0.025	0.219	0.21	0.074	0.09	0.149	0.104
28-Jan-2002	19	5.1107	5.2	0.025013	0.212	0.204	0.068	0.083	0.145	0.103
29-Jan-2002	18	4.9477	5.2	0.025	0.137	0.128	0.021	0.028	0.08	0.053
30-Jan-2002	17	5.0032	5.2	0.0242	0.416	0.412	0.275	0.296	0.294	0.062

VI. CONCLUSION

In this chapter we developed four different models to represent the volatility: Random Walk, ARCH, GARCH, and TARCH. Then, we put these four models into a Monte Carlo simulation in order to estimate the values of a selection of European options. We found that TARCH is the best model for predicting the market prices of European options.

However, there are still limitations of this chapter. I only use options from one company (Morgan Stanley) in this chapter. I will choose different options in due course, and hope to reach similar conclusions. Moreover, I have only considered European options. Extending this methodology to American options is awkward because an early-exercise condition would need to be checked at every step of the simulation. This is another topic for further research in this thesis.

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