Nonlinear Mathematical Physics 1996, V.3, N 3–4, 421–425.

On Some Generalized Symmetric Integral Operators of Buschman-Erdelyi's Type

N. VIRCHENKO

Kyiv Polytechnical Institute, 252056, Prospect Peremogy, 37

Abstract

Some new symmetric integral operators with kernels involving the generalized Legendre's function of the first kind $P_k^{m,n}(z)$ are introduced. Some their applications are given.

The last time the integral transforms with more complicated special functions in the kernels (G-, H-functions) are calling a great interest [1–3]. Exploring such integral transforms gives a possibility to deeper uncover the nature and character of integral transforms with simple kernels.

Buschman-Erdelyi's integral operators have the following form [4–5]:

$$Bf(x) = \int_0^x (x^2 - t^2)^{-\mu/2} P_\nu^\mu \left(\frac{x}{t}\right) f(t)dt, \tag{1}$$

$$Bf(x) = \int_0^x (x^2 - t^2)^{-\mu/2} P_\nu^\mu \left(\frac{t}{x}\right) f(t)dt \tag{2}$$

where $P_\nu^\mu(z)$ is the Legendre's function of the first kind, $f(x)$ is a locally summable function and satisfies necessary conditions as $x \to 0, x \to \infty; \mu, \nu$ are complex numbers, $\operatorname{Re}\mu < 1$, $\operatorname{Re}\nu \geq -1/2$. Let us notice that these operators are also known with the integral limites from x to ∞ [4–5].

The operators of such type are important for mathematical physics (in the solving of Dirichlet's problem for the Euler-Poisson-Darboux equation in the quadrant-plane [6], in the theory of Radon's transform [7], [8], in the theory of the elliptic equations with singular points [9], etc.

We shall consider some integral operators with the generalized associated Legendre's functions $P_k^{m,n}(z)$.

The generalized associated functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ are two linear-independent solutions of the following differential equation [10]:

$$(1 - z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left[k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right] u = 0 \tag{3}$$

where k, m, n can be complex in the general case. These solutions give rise to the definition of a class of functions $P_k^{m,n}(z), Q_k^{m,n}(z)$, which for $m = n$ are the same as the well-known associated Legendre's functions $P_k^{m,n}(z), Q_k^{m,n}(z)$, respectively.

Functions $P_k^{m,n}(z), Q_k^{m,n}(z)$ belong to the class of hypergeometric functions. Some integral representations for $P_k^{m,n}(z), Q_k^{m,n}(z)$ are established in [11].

The functions $P_k^{m,n}(z), Q_k^{m,n}(z)$ are arised in the solving of the sufficiently wide class of the boundary value problems of mathematical physics, mechanics of continuous medium, etc., in the different more complicated systems of orthogonal coordinates (ellipsoidal, toroidal, bipolar, spherical, etc.).

Let us introduce the following integral operator:

$$P(f)(x) = \int_0^\infty (t+x)^{\frac{n}{2}} (t-x)^{\frac{m}{2}} P_k^{m,n}\left(\frac{t}{x}\right) f(t)dt \tag{4}$$

where $\mathrm{Re}\, m < \frac{1}{2}$, $\frac{3}{2}m - \frac{n}{2} - 1 < k < \frac{-(m+n)}{2}, x > 0, f \in L_p(0,\infty), 1 < p < \infty$, $P_k^{m,n}(z)$ is the generalized associated Legendre's function.

Theorem 1 *If* $k + \frac{m+n}{2} > -1$, $m < n < m+1$, $\frac{3}{2}m - \frac{n}{2} - 1 < k < -\frac{(m+n)}{2}$, $\mathrm{Re}\, m < \frac{1}{2}$, *then the kernel of the integral operator (4) has the following integral representation:*

$$(t+x)^{\frac{n}{2}} (t-x)^{\frac{m}{2}} P_k^{m,n}\left(\frac{t}{x}\right) = C x^{\frac{n-m+1}{2}} \times \tag{5}$$

$$\int_0^\infty e^{-t\varepsilon} \varepsilon^{-m-\frac{1}{2}} K_{k-\frac{m-n}{2}+\frac{1}{2}}(x\varepsilon) {}_1F_1(n-m; k - \frac{3}{2} + \frac{n}{2} + 1; (t-x)\varepsilon) d\varepsilon,$$

where

$$C = \sqrt{\frac{2}{\pi}} 2^{n-m} \Gamma^{-1}\left(-k - \frac{m+n}{2}\right) \Gamma^{-1}\left(k - \frac{3}{2}m + \frac{n}{2} + 1\right),$$

K_ν is the modified Bessel function, ${}_1F_1(a;c;z) = \Phi(a;c;z)$ is the confluent hypergeometric function.

Proof. Using the integral representation for $K_\nu(xt)$ [12], the result of application of Laplace's integral transform to the function K_ν [13], we arrive at:

$$P_\nu^{-\mu}\left(\frac{t}{x}\right) = \sqrt{\frac{2x}{\pi}} \frac{(t^2 - x^2)^{\mu/2}}{\Gamma(\mu - \nu)\Gamma(\mu + \nu + 1)} \int_0^\infty e^{-t\varepsilon} \varepsilon^{\mu - 1} K_{\nu + \frac{1}{2}}(x\varepsilon) d\varepsilon, \tag{6}$$

$$\mathrm{Re}\,\nu > -\frac{1}{2}, \quad \mathrm{Re}\,(\mu - \nu) > 0, \quad \mathrm{Re}\,(\mu + \nu) > -1, \quad \mathrm{Re}\, t > 0.$$

According with [10], the function $P_k^{m,n}(z)$ can be represented in the form:

$$P_k^{m,n}(z) = \frac{2^{n-1}(z+1)^{\frac{m-n}{2}} (z^2-1)^{\frac{m}{2}}}{i\sqrt{\pi}\Gamma(\frac{1}{2} - m)\cos \pi m} \int_0^{(a+, \frac{1}{a}-} \varepsilon^{k + \frac{m+n}{2}} (1 - 2z\varepsilon + \varepsilon^2)^{-m-\frac{1}{2}} \times$$

$$ {}_2F_1\left(-k-\frac{m+n}{2}, n-m;\ \frac{1}{2}-m; \frac{-(\varepsilon^2-2z\varepsilon+1)}{2\varepsilon(z+1)}\right) d\varepsilon, \tag{7} $$

where the integral is written in the notations of Pochgammer. If $a = z+\sqrt{z^2-1}$ and the contour of integration is such that $\mid arg\varepsilon \mid < \pi$, then $P_k^{m,n}(z)$ can be expressed in terms of usual Legendre's function $P_k^m(z)$ [10].

Taking into account of (6),(7), we arrive at:

$$ P_k^{m,n}\left(\frac{t}{x}\right) = \sqrt{\frac{2x}{\pi}}2^{n-m}\left(\frac{t+x}{x}\right)^{\frac{m-n}{2}}(t^2-x^2)^{-\frac{m}{2}}\times $$
$$ \Gamma(k+\frac{m+n}{2}+1)\Gamma(m-n+1)\int\limits_0^\infty e^{-t\varepsilon}\varepsilon^{-m-\frac{1}{2}}K_{k-\frac{m-n-1}{2}}(x\varepsilon)\times $$
$$ \left(\sum_{i=0}^{\infty}(t^2-x^2)^{\frac{i}{2}}\varepsilon^i\left(\frac{t-x}{t+x}\right)^{\frac{i}{2}}\right)\Gamma^{-1}\left(i-k-\frac{m+n}{2}\right)\times \tag{8} $$
$$ \Gamma^{-1}\left(i+k-\frac{3}{2}m+\frac{n}{2}+1\right)\Gamma^{-1}\left(k+\frac{m+n}{2}-i+1\right)d\varepsilon. $$

Having taken of the well-known formulae of the theory of special functions [14]:

$$ \frac{\Gamma(a)}{\Gamma(a-i)} = (-1)^i\,\frac{\Gamma(i-a+1)}{\Gamma(1-a)},\ (a)_i = \frac{\Gamma(a+i)}{\Gamma(a)}, $$

after some transformations we obtain

$$ P_k^{m,n}\left(\frac{t}{x}\right) = \sqrt{\frac{2}{\pi}}x^{\frac{1-m+n}{2}}2^{n-m}(t+x)^{-\frac{n}{2}}(t-x)^{-\frac{m}{2}}\times $$
$$ \Gamma^{-1}\left(-k-\frac{m+n}{2}\right)\Gamma^{-1}\left(k-\frac{3}{2}m+\frac{n}{2}+1\right)\int\limits_0^\infty e^{-t\varepsilon}\varepsilon^{-m-\frac{1}{2}}\times \tag{9} $$
$$ K_{k-\frac{m-n}{2}+\frac{1}{2}}(x\varepsilon)\Phi\left(n-m; k-\frac{3}{2}m+1; (t-x)\varepsilon\right)d\varepsilon. $$

Hence (9) proves (5).

Corollary If $\mathrm{Re}\, m < \frac{1}{2}, \frac{3}{2}m-\frac{n}{2}-1 < k < -\frac{m+n}{2},\ x>0,\ f \in L_p(0,\infty),\ 1<p<\infty,$ then the integral operator (4) belongs to $L_p(0,\infty)$.

Further we introduce the following integral operator

$$ \tilde{P}f(x) = \int\limits_0^x (x-t)^{-\frac{m}{2}}(x+t)^{-\frac{n}{2}}P_k^{m,n}\left(\frac{x}{t}\right)f(t)dt \tag{10} $$

where $m<1m, n<1, \frac{m-n}{2}-1<k-(m+n)/2.$

Theorem 2 *If* $\alpha > 0, x \in [a;b], m < 1, n < 1,\ 0 < t < x,\ \frac{m-n}{2} - 1 < k < -\frac{m+n}{2}$, *then the kernel of the integral operator (10) has the following integral representation:*

$$(x-t)^{-\frac{m}{2}}(x+t)^{-\frac{n}{2}}P_k^{m,n}\left(\frac{x}{t}\right)H(x-t) = 2^{-k-\frac{m-n}{2}}t^{-k}I_x^{-k-\frac{m+n}{2}} \times$$

$$\left\{\frac{H(x-t)}{\Gamma(k-\frac{m-n}{2}+1)}(x^2-t^2)k + \frac{m-n}{2}(x-t)^{n-m}\right\}, \tag{11}$$

where I_x^{α} *is the fractional integral of Riemann–Lioville [1],* $H(x)$ *is a unit Heaviside function.*

As an example of application of the above results we give evaluation of some improper integrals with the special functions.

$$1)\quad \int_0^{\frac{\pi}{2}} (ch\beta + sh\beta\cos x)^{k+\frac{m+n}{2}}\sin^{-2m}x \times$$

$${}_2F_1\left(n-m, -k-\frac{m+n}{2}; \frac{1}{2}-m; \frac{\sinh^2\frac{\beta}{2}\sin^2 x}{\cosh\beta + \sinh\beta\cos x}\right)dx =$$

$$\sqrt{\pi}2^{-n}(\cosh\beta+1)^{\frac{n-m}{2}}\sinh^m\beta\Gamma\left(\frac{1}{2}-m\right)P_k^{m,n}(\cosh\beta) \tag{12}$$

$$(\operatorname{Re} m < \frac{1}{2}, k > -\frac{m+n}{2} - 1, 0 < n-m < 1).$$

$$2)\quad \int_0^{\infty} x^{-2m}[I_r(bx)Y_{-r}(cx) + I_{-r}(bx)Y_r(cx)] \times$$

$${}_2F_3(n-m, -k-\frac{m+n}{2}; \frac{1}{2}-m, -n-2k-\frac{1}{2}, \tag{13}$$

$$n-2m+2k+\frac{3}{2}; -\frac{x^2}{4}(b-c)^2)dx =$$

$$\frac{-2^{\frac{-m-3n}{2}}(\mid b-c\mid)^m(b+c)^n r(-n-2k-\frac{1}{2})}{\sqrt{\pi}(bc^{\frac{1+n-m}{2}}\Gamma(\frac{1}{2}+m)\Gamma^{-1}(n-2m+2k+\frac{3}{2})}P_k^{m,n}\left(\frac{b^2+c^2}{2bc}\right)$$

$$(r = m-n-2k-1, \mid m \mid < \frac{1}{2}, n > m, k < -\frac{m+n}{2},$$

$$n-2m+2k+\frac{3}{2} > 0; b, c > 0).$$

References

[1] Samko S., Kilbas A., Marichev O., Integrals and Derivatives of Fractional Order and Some its Applications, Minsk, 1987, 688p. (in Russian).

[2] Gupta K.C., Mittal P.K., The H-function transform, *J. of Austr. Math. Soc.,* 1970, V.11, 2, 142–148.

[3] Rooney P.G., On integral transformations with G-function kernels, *Proc. Royal Soc. of Edinburgh A,* 1983, V.93, N 3–4, 181–188.

[4] Buschman R.G., An inversion integral for a general Legendre transformation, *SIAM Rev.,* 1963, N 3, 232–233.

[5] Erdelyi A., An integral equations involving Legendre functions, *Soc. Industr. and Appl. Math.,* 1964, V.12, N 1, 15–30.

[6] Copson E.T., On a singular boundary value problem for an equation of hyperbolic type, *Arct. Ration. Mech. and Anal,* 1958, V.1, N4, 349–356.

[7] Lulwig D., The Radon transform on Euclidian Space, *Comm. on Pure and Appl.,* 1966, V.19, 49–81.

[8] Muller C., Richberg R. Über die Radon transformation, *Math. Meth. Appl. Sci.,* 1960, 106–109.

[9] Katrachov V., Singular boundary value problems and the operators of transformation, In: The correct boundary value problems for non-classic equations of math. physics, Novosibirsk, 1981, 87–91 (in Russian).

[10] Kuipers L., Meulenbeld B., On a generalization of Legendre's associated differential equation I,II, *Proc. Kon. Ned. Ak. v. W., Ser. A,* 1957, V.60, N 4. 337–350.

[11] Virchenko N., On some applications of the generalized associated Legendre's functions, *Ukr. math. J.,* 1987, V.39, N 2, 149–156 (in Russian).

[12] Watson H., The Theory of Bessel function, Moscow, 1949, I, 793p.

[13] Bateman H., Erdelyi A., The Tables of Integral Transforms, Moscow, 1969, I, 344p.

[14] Bateman H., Erdelyi A., Higher Transcendental Functions, Moscow, 1965, I, 296p.