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Symmetries of the Fokker-Type Relativistic Mechanics in Various Forms of Dynamics

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Abstract

The single-time nonlocal Lagrangians corresponding to the Fokker-type action integrals are obtained in arbitrary form of relativistic dynamics. The symmetry conditions for such Lagrangians under an arbitrary Lie group acting on the Minkowski space are formulated in various forms of dynamics. An explicit expression for the integrals of motion corresponding to the Poincaré invariance is derived.

Introduction

There are various approaches to relativistic direct-particle-interaction theory [1–3]. This theory does not use the notion of fields as mediators of interaction, but considers only particle degrees of freedom to be physically meaningful. The Fokker-type relativistic mechanics [4–6] which is the oldest attempt to construct such a theory has direct relation to the field description. It is based on the manifestly Poincaré-invariant, variational principle formulated in terms of four-dimensional coordinates and velocities of particles. Such a variational principle was first introduced for the electromagnetic interaction by Schwarzshild, Tetrode, and Fokker at the beginning of the century and was developed by various authors (see references in [1, 5, 6]). Later this description was extended on other relativistic interactions [6, 7]. The equations of motion following from such a variational principle explicitly answer the demand of relativistic invariance and can be compared with corresponding field theory expressions. However, this approach is not free of difficulties both on physical and mathematical levels. The cost for a manifestly Poincaré-invariant four-dimensional description is the necessity to use the many-time formalism, which complicates the physical interpretation of its results. Mathematically, it is hard to motivate obtaining the equations of motion from the action integrals which are obviously divergent because integration is carried out on the whole of the world lines of the particles [4].

It was showed in [8, 9] that many-time Fokker-type action integrals can be transformed into single-time actions with nonlocal Lagrangians depending on the three-dimensional coordinates of particles and on derivatives of the coordinates with respect to parameter t. Such Lagrangians provide us with a useful tool for the consideration of the various approximations [8–10] as well as for the transition to the predictive relativistic mechanics and Hamiltonian formalism [11,12]. It was demonstrated [9] that nonlocal Lagrangians corresponding to the manifestly Poincaré-invariant action integrals satisty the Poincaré-invariance conditions within the framework of a three-dimensional Lagrangian description of interacting particle systems [13]. The conservation laws which follow from such invariance were investigated via Noether's theorem. Moreover, as was stressed in [9], the nonlocal single-time Lagrangians which are found on the basis of the Fokker-type action integrals represent a close form for a wide class of solutions of the equations expressing the requirement of invariance of a Lagrangian description of particle systems underthe Poincaré group.

The purpose of this report is to study possible generalizations of this development. In the previous papers [8, 9] the evolution parameter t was chosen as coordinate time x^0. This choice corresponds to the instant form of relativistic dynamics in the Dirac's terminology [14]. Not long ago the single-time Lagrangian description of particle systems was extended to an arbitrary form of relativistic dynamics defined geometrically by means of the space-like foliations of the Minkowski space [15]. The conditions of the Poincaré-invariance were reformulated in an arbitrary form of dynamics and the expressions for corresponding conserved quantitites were obtained. Here we shall show how one can present solutions of Poincaré-invariance conditions in an arbitrary form of Lagrangian dynamics for the interactions originally described by a Fokker-type action.

In Sec.1 we derive a single-time Lagrangian function corresponding to the Fokker-type action integral in an arbitrary form of relativistic dynamics. The Poincaré-invariance conditions for such a Lagrangian are examined in Sec.2. Finally, in Sec.3 we find an explicit expression for conserved quantities in the terms of interaction potential functions entering Fokker-type integrals.

1 Fokker-type action integrals in an arbitrary form of relativistic dynamics

We shall be concerned with a dynamical system consisting of N interacting point particles. It is convenient to describe the evolution of this system in the four-dimensional Minkowski space M_4 with coordinates $x^\mu, \mu = 0, 1, 2, 3$. We use the metric $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. The motion of the particles is described by the world lines $\gamma_a : \mathrm{R} \to \mathrm{M}_4, a = 1, ..., N$, which can be parametrized by arbitrary parameters τ_a. In coordinates we have

$$\gamma_a : \tau_a \mapsto x_a^\mu(\tau_a). \tag{1.1}$$

We assume that the equations of motion for the system can be obtained from the Fokker-type variational principle

$$\delta S = 0, \tag{1.2}$$

where the action S has the form

$$S = \sum_a \int d\tau_a \Lambda_a(x_a, u_a) + \sum_{a<} \sum_b \int d\tau_a \int d\tau_b \Lambda_{ab}(x_a, x_b, u_a, u_b). \tag{1.3}$$

The functions Λ_a and Λ_{ab} depend on the four-dimensional particle coordinates x_a^μ and on the first derivatives

$$u_a^\mu = \frac{dx_a^\mu}{d\tau_a}. \tag{1.4}$$

It is well known that physical information about motion of the system is contained in the world lines γ_a considered as unparametrized paths in the Minkowski space. Therefore, the action (1.3) is assumed to be parameter-invariant. This assumption leads to the following conditions:

$$u_a^\mu \frac{\partial \Lambda_a}{\partial u_a^\mu} = \Lambda_a, \qquad u_a^\mu \frac{\partial \Lambda_{ab}}{\partial u_a^\mu} + u_b^\mu \frac{\partial \Lambda_{ab}}{\partial u_b^\mu} = \Lambda_{ab}. \tag{1.5}$$

Since in the Poincaré–invariant theory any particle cannot move with a velocity larger than that of light c, the world lines γ_a must be time-like lines, and for the tangent vectors (1.4) we have the unequality

$$u_a^2 \equiv \eta_{\mu\nu} u_a^\mu u_a^\nu \equiv u_a \cdot u_a > 0. \tag{1.6}$$

Then a solution of the equations (1.5) can be written in the form

$$\Lambda_a = \sqrt{u_a^2} f_a(x_a, \hat{u}_a), \qquad \Lambda_{ab} = \sqrt{u_a^2 u_b^2} f_{ab}(x_a, x_b, \hat{u}_a, \hat{u}_b), \tag{1.7}$$

where $\hat{u}_a^\mu = u_a^\mu / \sqrt{u_a^2}$. In the case of the Poincaré–invariant theory it will be assumed additionally that the arguments of the functions (1.7) are translationally-invariant Lorentz scalars. This gives

$$f_a = \text{const} = -m_a c, \tag{1.8}$$

$$f_{ab} = -c^{-1} F_{ab}(\rho_{ab}, \sigma_{ab}, \sigma_{ba}, \omega_{ab}), \qquad a < b, \tag{1.9}$$

where m_a is a rest mass of the particle a, the constant c is introduced for convenience, and the set of two-body invariants in (1.9) is chosen in the form [7,16,8]

$$\rho_{ab} = (x_a - x_b)^2, \qquad \sigma_{ab} = \eta_{ab}(x_a - x_b) \cdot \hat{u}_a, \qquad \omega_{ab} = \hat{u}_a \cdot \hat{u}_b, \tag{1.10}$$

with $\eta_{ab} = sgn(b - a)$.

The concept of the forms of relativistic dynamics [14] may be introduced within the framework of the single-time three-dimensional Lagrangian formalism in the following way [15,10]. Let us consider the foliation Σ of the Minkowski space M_4 by the hypersurfaces

$$t = \sigma(x), \qquad t \in \mathrm{R}, \tag{1.11}$$

with the next property. Every hypersurface $\Sigma_t = \{x \in \mathrm{M}_4 | \sigma(x) = t\}$ must intersect the world lines γ_a of all particles in one and only one point

$$x_a(t) = \gamma_a \bigcap \Sigma_t. \tag{1.12}$$

This allows us to consider the t as an evolution parameter of the system [17,18,15]. In the Poincaré-invariant theory, when we consider only time-like world lines, the hypersurfaces (1.11) must be space-like or isotropic

$$\eta_{\mu\nu}(\partial^\mu \sigma)(\partial^\nu \sigma) \geq 0, \tag{1.13}$$

where $\partial^\mu = \partial / \partial x_\mu$. Then we have $\partial^0 \sigma > 0$, and the hypersurface equation (1.11) has the solution $x^0 = \varphi(t, \mathbf{x})$, where $\mathbf{x} = (x^i)$, $i = 1, 2, 3$. Therefore, the constraint $x_a(t) \in \Sigma_t$ enables us to determine a zero component of $x_a(t)$ in the terms of t and $x_a^i(t)$, $i = 1, 2, 3$.

The parametric equations (1.1) of the world lines of particles in the given form of dynamics have the form

$$x^0 = \varphi(t, \mathbf{x}_a(t)) \equiv \varphi_a, \qquad x^i = x^i_a(t). \tag{1.14}$$

The evolution of the system is determined by $3N$ functions $t \mapsto x^i_a(t)$. They may be considered as representatives for the sections $s : \mathrm{R} \to \mathrm{F}, t \mapsto (t, x^i_a(t))$ of the trivial fibre bundle $\pi : \mathrm{F} \to \mathrm{R}$ with a 3N-dimensional fibre space $\mathrm{E} = \mathrm{R}^{3N}$. The latter constitutes the configuration space of our system.

Three Dirac's forms of relativistic dynamics correspond to the following hypersurfaces (1.11): $x^0 = ct$ (instant form), $x^0 - x^3 = ct$ (front form), and $x \cdot x = c^2t^2$ (point form). Other examples may be found in [15].

Now we assume that the evolution of the system under consideration is completely determined by specifying an action functional

$$S = \int dtL. \tag{1.15}$$

The Lagrangian function $L : J^\infty\pi \to \mathrm{R}$ is defined on the infinite-order jet space of the fibre bundle $\pi : \mathrm{F} \to \mathrm{R}$ with the standard coordinates $x^{i(s)}_a$ [22–24]. The values of these coordinates for the section $s : t \mapsto (t, x^i_a(t))$ belonging to the corresponding equivalence class from $J^\infty\pi$ are $x^{i(s)}_a(t) = d^s x^i_a(t)/dt^s, s = 0, 1, 2, ...$ The variational principle (1.2) with action (1.15) gives Euler-Lagrange equations of motion

$$\mathcal{E}_{ai}L \equiv \sum_{s=0}^{\infty}(-D)^s \frac{\partial L}{\partial x^{i(s)}_a} = 0, \tag{1.16}$$

where D is an operator of the total time derivative

$$D = \sum_a \sum_{s=0}^{\infty} x^{i(s+1)}_a \frac{\partial}{\partial x^{i(s)}_a} + \frac{\partial}{\partial t}. \tag{1.17}$$

Now we consider the transition from Fokker-type action (1.3) to its single-time counterpart (1.15) in an arbitrary form of relativistic dynamics. Because of parametric invariance of the action (1.3) we can choose the parameters τ_a to be equal to t_a such that $x_a(t_a) \in \Sigma_{t_a}$. This allows us to determine zero components of the four-vectors $x_a(t_a)$ and $u_a(t_a)$ in the terms of three-dimensional quantities:

$$x^0_a(t_a) = \varphi(t_a, \mathbf{x}_a(t_a)) = \varphi_a(t_a), \tag{1.18}$$

$$u^0_a(t_a) = \frac{d}{dt_a}\varphi(t_a, \mathbf{x}_a(t_a)) = D\varphi_a(t_a). \tag{1.19}$$

Thus

$$\hat{u}_a = c^{-1}\Gamma_a(t_a)(D\varphi_a(t_a), \mathbf{v}_a(t_a)), \tag{1.20}$$

where $\mathbf{v}_a(t_a) = d\mathbf{x}_a(t_a)/dt_a$ and

$$c\Gamma^{-1}_a(t) = \sqrt{(D\varphi_a(t))^2 - v^2_a(t)} = \sqrt{u^2_a(t)}. \tag{1.21}$$

The following transformations are quite similar to that used in paper [8] within the framework of the instant form of dynamics. Using (1.7)–(1.9) and (1.21) in (1.3), we obtain

$$\begin{aligned} S &= -\sum_a m_a c^2 \int dt_a \Gamma_a^{-1}(t_a) \\ &- c\sum_{a<}\sum_b \int dt_a \int dt_b \Gamma_a^{-1}(t_a)\Gamma_b^{-1}(t_b) F_{ab}(x_a(t_a), x_b(t_b), \hat{u}_a(t_a), \hat{u}_b(t_b)). \end{aligned} \tag{1.22}$$

Identifying the integration variables in each term of the sums in (1.22) gives

$$S = -\sum_a m_a c^2 \int dt_a \Gamma_a^{-1}(t) - c\sum_{a<}\sum_b \int dt_1 \int dt_2 \chi_{ab}[t_1, t_2], \tag{1.23}$$

where

$$\chi_{ab}[t_1, t_2] = \Gamma_a^{-1}(t_1)\Gamma_b^{-1}(t_2) F_{ab}(x_a(t_1), x_b(t_2), \hat{u}_a(t_1), \hat{u}_b(t_2)). \tag{1.24}$$

Changing the integration variables in the double sums of (1.23), $(t_1, t_2) \mapsto (t, \theta)$, where $t = \lambda t_1 + (1-\lambda)t_2, \theta = c(t_1 - t_2)$, and $\lambda \in \mathrm{R}$ is an arbitrary number (see [19,8]), we get the expression (1.15) with the Lagrangian function

$$L = L_f - U, \tag{1.25}$$

where

$$L_f = -\sum_a m_a c^2 \Gamma_a^{-1}(t) \tag{1.26}$$

is a free-particle Lagrangian in an arbitrary form of dynamics and

$$U = \sum_{a<}\sum_b \int d\theta \chi_{ab}\left(t + (\lambda - 1)\frac{\theta}{c}, t + \lambda\frac{\theta}{c}\right) \tag{1.27}$$

is an interaction potential. The shifts of the time arguments in (1.27) may be expressed by means of the exponential operators $\exp(\alpha D_a)$, where

$$D_a = \sum_{s=0}^{\infty} x_a^{i(s+1)} \frac{\partial}{\partial x_a^{i(s)}}. \tag{1.28}$$

Using the obvious relation

$$\sum_a D_a = D - \frac{\partial}{\partial t} \tag{1.29}$$

and commutativity of the operators D_a and D_b, we obtain

$$U = \sum_{a<}\sum_b \int d\theta \exp\left(\frac{\theta}{c}(\lambda D - D_a)\right) \chi_{ab}\left(t - \frac{\theta}{c}, t, \mathbf{x}_a, \mathbf{x}_b, \mathbf{v}_a, \mathbf{v}_b\right). \tag{1.30}$$

It should be noted that an arbitrary parameter λ enters the Lagrangian function (1.30) only together with the operator of a total time derivative D and, therefore, has no effect on the observable quantities.

In the case of Poincaré-invariant interactions determined by the relation (1.9), we have

$$\chi_{ab}[t_1, t_2] = (\Gamma'_a)^{-1}\Gamma_b^{-1} F_{ab}(\rho_{ab}, \sigma_{ab}, \sigma_{ba}, \omega_{ab}), \tag{1.31}$$

where the arguments (1.10) of the functions F_{ab} are expressed in the three-dimensional terms ($a < b$)

$$\rho_{ab} = (\varphi'_a - \varphi_b)^2 - r_{ab}^2, \tag{1.32}$$

$$\sigma_{ab} = c^{-1}\Gamma'_a[(\varphi'_a - \varphi_b)D\varphi'_a - \mathbf{r}_{ab}\cdot\mathbf{v}_a], \tag{1.33}$$

$$\sigma_{ba} = c^{-1}\Gamma_b[(\varphi'_a - \varphi_b)D\varphi_b - \mathbf{r}_{ab}\cdot\mathbf{v}_b], \tag{1.34}$$

$$\omega_{ab} = c^{-2}\Gamma'_a\Gamma_b[(D\varphi'_a)D\varphi_b - \mathbf{v}_a\cdot\mathbf{v}_b], \tag{1.35}$$

and the following abbreviations are used

$$\mathbf{r}_{ab} \equiv \mathbf{x}_a - \mathbf{x}_b, \qquad \varphi'_a \equiv \exp\left(-\frac{\theta}{c}\frac{\partial}{\partial t}\right)\varphi_a, \qquad \Gamma'_a \equiv \exp\left(-\frac{\theta}{c}\frac{\partial}{\partial t}\right)\Gamma_a. \tag{1.36}$$

For the instant form of dynamics, when

$$\sigma(x) = c^{-1}x^0, \qquad \varphi(t, \mathbf{x}) = ct, \tag{1.37}$$

we return to the expressions of the paper [9]:

$$\Gamma_a = (1 - v_a^2 c^{-2})^{-1/2} \equiv \gamma_a, \tag{1.38}$$

$$\rho_{ab} = \theta^2 - r_{ab}^2, \quad \sigma_{ab} = -\gamma_a(\theta + \eta_{ab}\mathbf{r}_{ab}\cdot\mathbf{v}_a c^{-1}), \quad \omega_{ab} = \gamma_a\gamma_b(1 - \mathbf{v}_a\cdot\mathbf{v}_b c^{-2}). \tag{1.39}$$

As another interesting example we consider the class of isotropic forms of dynamics determined by the hyperplanes

$$x^\mu n_\mu = ct, \qquad n^\mu n_\mu = 0, \tag{1.40}$$

with a null vector $n^\mu = (1, \mathbf{n}), |\mathbf{n}| = 1$. In this case we have

$$\varphi(t, \mathbf{x}) = ct + \mathbf{n}\cdot\mathbf{x}, \tag{1.41}$$

and

$$\Gamma_a = [(1 + \mathbf{n}\cdot\mathbf{v}_a c^{-1})^2 - v_a^2 c^{-2}]^{-1/2}. \tag{1.42}$$

For the invariants (1.32)–(1.35) in these forms of dynamics, we find

$$\rho_{ab} = (\theta - \mathbf{n}\cdot\mathbf{r}_{ab})^2 - r_{ab}^2, \tag{1.43}$$

$$\sigma_{ab} = -\Gamma_a[(\theta - \eta_{ab}\mathbf{n}\cdot\mathbf{r}_{ab})(1 + \mathbf{n}\cdot\mathbf{v}_a c^{-1})) + \eta_{ab}\mathbf{r}_{ab}\cdot\mathbf{v}_a c^{-1}], \tag{1.44}$$

$$\omega_{ab} = \Gamma_a\Gamma_b[(1 + \mathbf{n}\cdot\mathbf{v}_a c^{-1})(1 + \mathbf{n}\cdot\mathbf{v}_b c^{-1}) - \mathbf{v}_a\cdot\mathbf{v}_b c^{-2}]. \tag{1.45}$$

Within the framework of relativistic Lagrangian mechanics such forms of dynamics have the most interesting applications in the two-dimensional space-time M_2, when ($n = \pm 1$)

$$\Gamma_a = (1 + 2nv_a c^{-1})^{-1/2}, \tag{1.46}$$

$$\rho_{ab} = \theta^2 - 2\theta n r_{ab}, \quad \sigma_{ab} = -\gamma_a[\theta(1 + nv_a c^{-1}) - n\eta_{ab}r_{ab}], \quad \omega_{ab} = \Gamma_a^{-1}\Gamma_b - \Gamma_a^{-1}\Gamma_b. \tag{1.47}$$

Several models of the relativistic direct particle interactions in this form of dynamics have been considered in [18, 20, 21].

2 Symmetries in the Lagrangian description

Let us consider an arbitrary r-parametric Lie group G acting on M_4 by the point transformations $g : \mathrm{M}_4 \to \mathrm{M}_4$,

$$x^\mu \mapsto (gx)^\mu = x^\mu + \lambda^\alpha \zeta^\mu_\alpha + o(\lambda), \tag{2.1}$$

where λ^α, $\alpha = 1, ..., r$, are parameters of the group. The vector fields

$$\mathcal{X}_\alpha = \zeta^\mu_\alpha \partial_\mu \tag{2.2}$$

satisfy the commutational relations of the Lie algebra of the group G,

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = c^\gamma_{\alpha\beta} \mathcal{X}_\gamma, \qquad \alpha, \beta, \gamma = 1, \ldots, r, \tag{2.3}$$

with the structure constants $c^\gamma_{\alpha\beta}$.

The action (2.1) of the group G on M_4 can be easily extended on the world-lines γ_a by the rule

$$\gamma_a \mapsto g\gamma_a = \{gx | x \in Im\gamma_a\}. \tag{2.4}$$

But in the given form of dynamics the world lines γ_a are determined by the functions $t \mapsto x^i_a(t)$ or, in other words, by the sections s of the bundle π. Therefore, (2.4) induces an action of the group G on $J^\infty\pi$ by the Lie-Bäcklund transformations [22–24]. As was showed in [15], the generators of such transformations have the form

$$X_\alpha = \sum_a \sum_{s=0}^{\infty} (D^s \xi^i_{a\alpha}) \frac{\partial}{\partial x^{i(s)}_a}, \tag{2.5}$$

where

$$\xi^i_{a\alpha} = \zeta^i_{a\alpha} - v^i_a \eta_{a\alpha}, \tag{2.6}$$

and

$$\zeta^i_{a\alpha} = \zeta^i_\alpha(t, \mathbf{x_a}), \qquad \eta_{a\alpha} = (X_\alpha \sigma)(t, \mathbf{x_a}), \qquad v^i_a = x^{i(1)}_a. \tag{2.7}$$

The Lie-Bäcklund vector fields (2.5) obey the same commutational relations as (2.2),

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma, \tag{2.8}$$

and commute with a total time derivative (1.17)

$$[X_\alpha, D] = 0. \tag{2.9}$$

For the Poincaré group we have the following ten vector fields corresponding to its natural action on M_4:

$$\mathcal{X}^T_\mu = \partial_\mu, \tag{2.10}$$

$$\mathcal{X}^L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \tag{2.11}$$

with commutational relations

$$[\mathcal{X}^T_\mu, \mathcal{X}^T_\mu] = 0, \tag{2.12}$$

$$[\mathcal{X}^T_\mu, \mathcal{X}^L_{\rho\sigma}] = \eta_{\mu\rho}\mathcal{X}^T_\sigma - \eta_{\mu\sigma}\mathcal{X}^T_\rho, \tag{2.13}$$

$$[\mathcal{X}^L_{\mu\nu}, \mathcal{X}^L_{\rho\sigma}] = \eta_{\nu\rho}\mathcal{X}^L_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{X}^L_{\nu\rho} - \eta_{\mu\rho}\mathcal{X}^L_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{X}^L_{\mu\rho}. \tag{2.14}$$

Thus, we obtain the next realization of the Poincaré algebra in terms of Lie-Bäcklund vector fields (2.5)

$$X^T_\mu = \sum_a \sum_{s=0}^{\infty} D^s[\delta^i_\mu - v^i_a \sigma_{a\mu}] \frac{\partial}{\partial x^{i(s)}_a}, \tag{2.15}$$

$$X^L_{\mu\nu} = \sum_a \sum_{s=0}^{\infty} D^s[x_{a\mu}\delta^i_\nu - x_{a\nu}\delta^i_\mu - v^i_a(x_{a\mu}\sigma_{a\nu} - x_{a\nu}\sigma_{a\mu}] \frac{\partial}{\partial x^{i(s)}_a}, \tag{2.16}$$

where we must use (1.14) for elimination of x^0_a, and we denote

$$\sigma_{a\mu} \equiv (\partial_\mu \sigma)(t, \mathbf{x}_a). \tag{2.17}$$

Making use of the hypersurface equation (1.11), we find

$$\sigma_{a0} = (\partial\varphi_a/\partial t)^{-1} \equiv \varphi^{-1}_{at}, \tag{2.18}$$

$$\sigma_{ai} = -\varphi^{-1}_{at}(\partial\varphi_a/\partial x_{ai}) \equiv -\varphi^{-1}_{at}\varphi_{ai}. \tag{2.19}$$

It is convenient to introduce the vector fields

$$\mathcal{H} = -cX^T_0, \quad \mathcal{P}_i = X^T_i, \quad \mathcal{J}_i = -\frac{1}{2}\epsilon_{ijk}X^L_{jk}, \quad \mathcal{K} = c^{-1}X^L_{i0}, \tag{2.20}$$

obeying the following commutational relations

$$[\mathcal{H}, \mathcal{P}_i] = 0, \qquad [\mathcal{P}_i, \mathcal{P}_j] = 0, \qquad [\mathcal{H}, \mathcal{J}_i] = 0, \qquad [\mathcal{P}_i, \mathcal{J}_k] = -\epsilon_{ikl}\mathcal{P}_l, \tag{2.21}$$

$$[\mathcal{J}_i, \mathcal{J}_k] = -\epsilon_{ikl}\mathcal{J}_l, \qquad [\mathcal{K}_i, \mathcal{J}_k] = -\epsilon_{ikl}\mathcal{K}_l, \qquad [\mathcal{K}_i, \mathcal{K}_j] = c^{-2}\epsilon_{ijk}\mathcal{J}_k, \tag{2.22}$$

$$[\mathcal{H}, \mathcal{K}_i] = \mathcal{P}_i, \qquad [\mathcal{P}_i, \mathcal{K}_j] = \delta_{ij}c^{-2}\mathcal{H}. \tag{2.23}$$

Inserting (2.18), (2.19) into (2.15), (2.16), we obtain the realization of the Poincaré algebra which is convenient for consideration of symmetries of the single-time three-dimensional Lagrangian description [15]:

$$\mathcal{H} = c\sum_a \sum_{s=0}^{\infty} D^s[v^i_a \varphi^{-1}_{at}] \frac{\partial}{\partial x^{i(s)}_a}, \tag{2.24}$$

$$\mathcal{P}_i = \sum_a \sum_{s=0}^{\infty} D^s[\delta^j_i + v^j_a \varphi_{ai}\varphi^{-1}_{at}] \frac{\partial}{\partial x^{j(s)}_a}, \tag{2.25}$$

$$\mathcal{J}_i = \epsilon_{ikl}\sum_a \sum_{s=0}^{\infty} D^s[x^k_a(\delta^j_l + v^j_a \varphi_{al}\varphi^{-1}_{at})] \frac{\partial}{\partial x^{j(s)}_a}, \tag{2.26}$$

$$\mathcal{K}_i = c^{-1}\sum_a \sum_{s=0}^{\infty} D^s[-\varphi_a\delta_i^j + v_a^j(x_{ai} - \varphi_a\varphi_{ai})\varphi_{at}^{-1}]\frac{\partial}{\partial x_a^{j(s)}}. \tag{2.27}$$

The symmetry of a Lagrangian description of the interacting particle system under the group G means the invariance of the Euler-Lagrange equation (1.16) under corresponding Lie-Bäcklund transformations generated by the vector fields (2.5). It was showed and discussed in [13,10] that the sufficient conditions for the symmetry under Poincaré group have the form

$$X_\alpha L = D\Omega_\alpha, \qquad \alpha = 1,\cdots,10, \tag{2.28}$$

with auxiliary functions Ω_α satisfying the consistency relations

$$X_\alpha\Omega_\beta - X_\beta\Omega_\alpha = c_{\alpha\beta}^\gamma\Omega_\gamma. \tag{2.29}$$

Let us examine these conditions for the nonlocal Lagrangians corresponding to the manifestly Poincaré-invariant Fokker-type action. To do it, we need the following formula for derivatives of the interaction potential (1.30) [19,8]

$$\begin{aligned}\frac{\partial U}{\partial x_a^{i(s)}} &= \int d\theta \exp\left(\frac{\theta}{c}\lambda D\right)\left\{\sum_{b(>a)} \exp\left(-\frac{\theta}{c}D_a\right)\left[\frac{(\lambda-1)^s\theta^s}{c^s s!}\frac{\partial\chi_{ab}}{\partial x_a^i} + \right.\right.\\ &\left.\frac{(\lambda-1)^{s-1}\theta^{s-1}}{c^{s-1}(s-1)!}\frac{\partial\chi_{ab}}{\partial v_a^i}(1-\delta_{0s})\right] + \\ &\left.\sum_{b(<a)} \exp\left(-\frac{\theta}{c}D_b\right)\left[\frac{\lambda^s\theta^s}{c^s s!}\frac{\partial\chi_{ba}}{\partial x_a^i} + \frac{\lambda^{s-1}\theta^{s-1}}{c^{s-1}(s-1)!}\frac{\partial\chi_{ba}}{\partial v_a^i}(1-\delta_{0s})\right]\right\}. \end{aligned} \tag{2.30}$$

Taking this expression, we obtain after some calculations that nonlocal Lagrangians determined by the formulae (1.25)–(1.27) for the case of Poincaré-invariant interactions (1.31), (1.32) indeed satisfy the conditions (2.26), (2.27) with the following functions $\Omega_\alpha, \alpha = 1, ..., 10$:

$$\begin{aligned}\Omega_\alpha &= \sum_a m_a c^2 \eta_{a\alpha}\Gamma_a^{-1} + \sum_{a<}\sum_b \int d\theta \exp\left(\frac{\theta}{c}(\lambda D - D_a)\right)\chi_{ab} \times \\ &\left[\lambda \exp(-\frac{\theta}{c}\frac{\partial}{\partial t})\eta_{a\alpha} + (1-\lambda)\eta_{b\alpha}\right]. \end{aligned} \tag{2.31}$$

This conclusion extends the result of papers [8,9] on an arbitrary form of relativistic dynamics.

3 Conservation laws for nonlocal Lagrangians

The important corollary of symmetry conditions (2.28), (2.29) for an arbitrary r-parametric Lie group is the existence of r conservation laws

$$DG_\alpha = 0, \qquad \alpha = 1, ..., r, \tag{3.1}$$

for the quantities G_α which can be explicitly determined in terms of the Lagrangian function L and auxiliary functions Ω_α. This statement, which is well known as the Noether's theorem, follows immediately from the identity [22,23]

$$X_\alpha L = \sum_a \xi^i_{a\alpha} \mathcal{E}_{ai} L + D \sum_a \sum_{s=o}^{\infty} \pi_{ai,s} D^s \xi^i_{a\alpha}, \tag{3.2}$$

which holds for an arbitrary Lie-Bäcklund vector field (2.5). Here

$$\pi_{ai,s} = \sum_{n=s}^{\infty} (-D)^{n-s} \frac{\partial L}{\partial x_a^{i(n+1)}} \tag{3.3}$$

are Ostrogradskyj's momenta. Making use of the identity (3.2) in the symmetry conditions (2.28), one readily checks that for the solutions of Euler-Lagrange equation (1.16) the conservation laws (3.1) hold with

$$G_\alpha = \sum_a \sum_{s=o}^{\infty} \pi_{ai,s} D^s \xi^i_{a\alpha} - \Omega_\alpha. \tag{3.4}$$

In this section we evaluate explicitly the conserved quantities (3.4) corresponding to the Lagrangian function (1.25) which obeys the symmetry conditions (2.28), (2.29). Firstly, we shall consider the Ostrogradskyj's momenta $\pi_{ai,s}$. We make use of the formula (2.30) to bring (3.3) to the form

$$\begin{aligned} \pi_{ai,s} \quad = \quad & \frac{\partial L_f}{\partial v_a^i} \delta_{0s} - \int d\theta \exp\left(\frac{\theta}{c}\lambda D\right) \Bigg\{ \sum_{b(>a)} \exp\left(-\frac{\theta}{c} D_a\right) \Bigg[\frac{(\lambda-1)^s \theta^s}{c^s s!} \frac{\partial \chi_{ab}}{\partial v_a^i} + \\ & \sum_{n=0}^{\infty} \frac{(\lambda-1)^{n+s+1} \theta^{s+1}}{c^{n+s+1}(n+s+1)!} (-\theta D)^n \mathcal{E}_{ai} \chi_{ab} \Bigg] + \sum_{b(<a)} \exp\left(-\frac{\theta}{c} D_b\right) \Bigg[\frac{\lambda^s \theta^s}{c^s s!} \frac{\partial \chi_{ba}}{\partial v_a^i} + \\ & \sum_{n=0}^{\infty} \frac{\lambda^{n+s+1} \theta^{s+1}}{c^{n+s+1}(n+s+1)!} (-\theta D)^n \mathcal{E}_{ai} \chi_{ba} \Bigg] \Bigg\}. \end{aligned} \tag{3.5}$$

The equality

$$\frac{1}{(n+s+1)!} = \frac{1}{n!s!} \int_0^1 d\tau (1-\tau)^s \tau^n \tag{3.6}$$

allows us to perform the formal summation over n with the result

$$\begin{aligned} \pi_{ai,s} = & \sum_a m_a \Gamma_a (v_{ai} - \varphi_{ai} D\varphi_a) \delta_{0s} - \int d\theta \exp\left(\frac{\theta}{c}\lambda D\right) \Bigg\{ \sum_{b(>a)} \exp\left(-\frac{\theta}{c} D_a\right) \times \\ & \left[\frac{\partial \chi_{ab}}{\partial v_a^i} + \frac{(\lambda-1)\theta}{c} \int_0^1 d\tau \exp\left(\frac{\theta}{c}(\tau(1-\lambda) D\right) \mathcal{E}_{ai} \chi_{ab} (1-\tau)^s \right] \frac{(\lambda-1)^s \theta^s}{c^s s!} + \\ & \sum_{b(<a)} \exp\left(-\frac{\theta}{c} D_b\right) \left[\frac{\partial \chi_{ba}}{\partial v_a^i} + \frac{\lambda \theta}{c} \int_0^1 d\tau \exp\left(-\frac{\theta}{c} \tau \lambda D\right) \mathcal{E}_{ai} \chi_{ba} (1-\tau)^s \right] \frac{\lambda^s \theta^s}{c^s s!} \Bigg\}, \end{aligned} \tag{3.7}$$

where expression (1.26) for the free-particle Lagrangian L_f has been used.

Inserting (3.7) and (2.31) into (3.4) and summing over s, we obtain

$$\begin{aligned}
G_\alpha &= \sum_a m_a\Gamma_a[(v_{ai}-\varphi_{ai}D\varphi_a)\xi^i_{a\alpha}-c^2\Gamma_a^{-2}\eta_{a\alpha}]- \\
&\sum_{a<}\sum_b\int d\theta\exp\left(\frac{\theta}{c}(\lambda D-D_a)\right)\left[\frac{\partial\chi_{ab}}{\partial v^i_a}\exp\left(\frac{\theta}{c}\frac{\partial}{\partial t}\right)\xi^i_{a\alpha}+\frac{\partial\chi_{ab}}{\partial v^i_b}\xi^i_{b\alpha}+\right. \\
&\chi_{ab}\left(\lambda\exp\left(-\frac{\theta}{c}\frac{\partial}{\partial t}\right)\eta_{a\alpha}+(1-\lambda)\eta_{b\alpha}\right)\Bigg]+ \\
&\frac{(\lambda-1)\theta}{c}\int_0^1 d\tau\exp\left(\frac{\theta}{c}(\tau(1-\lambda)D\right)(\mathcal{E}_{ai}\chi_{ab})\exp\left(-\frac{\theta}{c}\frac{\partial}{\partial t}\right)\xi^i_{a\alpha}+ \\
&\left.\frac{\lambda\theta}{c}\int_0^1 d\tau\exp\left(-\frac{\theta}{c}\tau\lambda D\right)(\mathcal{E}_{bi}\chi_{ba})\xi^i_{b\alpha}\right].
\end{aligned} \tag{3.8}$$

Using in (3.8) the definition (2.6) of components of the Lie-Bäcklund fields (2.5) and taking into account the following relations

$$\zeta^0_{a\alpha}\equiv\zeta^0_\alpha(t,\mathbf{x}_a)=\varphi_{ai}\zeta^i_{a\alpha}+\varphi_{at}\eta_{a\alpha}, \tag{3.9}$$

$$\alpha\int_0^1 d\tau e^{\alpha\tau D}D=e^{\alpha D}-1, \tag{3.10}$$

$$\int d\theta\exp\left(\frac{\theta}{c}(\lambda D-D_a)\right)D_a=\int d\theta\exp\left(\frac{\theta}{c}(\lambda D-D_a)\right)\left(\lambda D+c\frac{\partial}{\partial\theta}\right), \tag{3.11}$$

$$\int d\theta\exp\left(\frac{\theta}{c}(\lambda D-D_a)\right)D_b=\int d\theta\exp\left(\frac{\theta}{c}(\lambda D-D_a)\right)\left((1-\lambda)D-c\frac{\partial}{\partial\theta}-\frac{\partial}{\partial t}\right) \tag{3.12}$$

(the two latter equalities are valid under the action on an arbitrary two-particle expression such as appears in the double sum in (3.8)), we can write down the explicit formula for the conserved quantities

$$\begin{aligned}
G_\alpha &= \sum_a m_a\Gamma_a(\zeta^i_{a\alpha}v_{ai}-\zeta^0_{a\alpha}D\varphi_a)- \\
&\sum_{a<}\sum_b\int d\theta\left\{\zeta^i_{a\alpha}\exp\left[\frac{\theta}{c}\left(D_b+\frac{\partial}{\partial t}\right)\right]\frac{\partial\chi_{ab}}{\partial v^i_a}+\zeta^i_{b\alpha}\exp\left(-\frac{\theta}{c}D_a\right)\frac{\partial\chi_{ab}}{\partial v^i_b}+\right. \\
&\eta_{a\alpha}\exp\left[\frac{\theta}{c}\left(D_b+\frac{\partial}{\partial t}\right)\right]\left(\chi_{ab}-v^i_a\frac{\partial\chi_{ab}}{\partial v^i_a}\right)+ \\
&\eta_{b\alpha}\exp\left(-\frac{\theta}{c}D_a\right)\left(\chi_{ab}-v^i_b\frac{\partial\chi_{ab}}{\partial v^i_b}\right)+ \\
&\frac{\theta}{c}\int_0^1 d\tau\exp\left(\frac{\theta}{c}(\tau D-D_a)\right)\left[\frac{\partial\chi_{ab}}{\partial x^i_b}\zeta^i_{b\alpha}+\frac{\partial\chi_{ab}}{\partial v^i_b}(D\zeta^i_{b\alpha}-v^i_bD\eta_{b\alpha})+\right. \\
&\left.\left.\left(\frac{\partial\chi_{ab}}{\partial t}+c\frac{\partial\chi_{ab}}{\partial\theta}\right)\eta_{b\alpha}+\chi_{ab}D\eta_{b\alpha}\right]\right\}.
\end{aligned} \tag{3.13}$$

We note that an arbitrary parameter λ is missing in the final formula for the conserved quantities. The same is also true for the particle equations of motion [8, 9].

The expressions for the components of the Lie-Bäcklund vector fields (2.24)–(2.27) corresponding to the Poincaré group can be used to obtain the conserved energy E, momentum $\mathbf{P}$, angular momentum $\mathbf{J}$, and center-of-mass integral of motion $\mathbf{K}$ in an arbitrary form of relativistic dynamics

$$\begin{aligned}
E &= c\sum_a m_a\Gamma_a D\varphi_a + \\
&c\sum_{a<}\sum_b\int d\theta\left\{\exp\left[\frac{\theta}{c}\left(D_b+\frac{\partial}{\partial t}\right)\right](\varphi'_{at})^{-1}\left(\chi_{ab}-v_a^i\frac{\partial\chi_{ab}}{\partial v_a^i}\right)+\right. \\
&\exp\left(-\frac{\theta}{c}D_a\right)\varphi_{bt}^{-1}\left(\chi_{ab}-v_b^i\frac{\partial\chi_{ab}}{\partial v_b^i}\right)+ \\
&\frac{\theta}{c}\int_0^1 d\tau\exp\left(\frac{\theta}{c}(\tau D-D_a)\right)\left[(D\varphi_{bt}^{-1})\left(\chi_{ab}-v_b^i\frac{\partial\chi_{ab}}{\partial v_b^i}\right)+\right. \\
&\left.\left.\varphi_{bt}^{-1}\left(\frac{\partial\chi_{ab}}{\partial t}+c\frac{\partial\chi_{ab}}{\partial\theta}\right)\right]\right\}, \qquad (3.14)
\end{aligned}$$

$$\begin{aligned}
P_i &= \sum_a m_a\Gamma_a v_a^i - \\
&\sum_{a<}\sum_b\int d\theta\left\{\exp\left[\frac{\theta}{c}\left(D_b+\frac{\partial}{\partial t}\right)\right]\left[\frac{\partial\chi_{ab}}{\partial v_a^i}-\varphi'_{ai}(\varphi'_{at})^{-1}\left(\chi_{ab}-v_a^j\frac{\partial\chi_{ab}}{\partial v_a^j}\right)\right]+\right. \\
&\exp\left(-\frac{\theta}{c}D_a\right)\left[\frac{\partial\chi_{ab}}{\partial v_b^i}-\varphi_{bi}\varphi_{bt}^{-1}\left(\chi_{ab}-v_b^j\frac{\partial\chi_{ab}}{\partial v_b^j}\right)\right]+ \\
&\frac{\theta}{c}\int_0^1 d\tau\exp\left(\frac{\theta}{c}(\tau D-D_a)\right)\left[\frac{\partial\chi_{ab}}{\partial x_b^i}-(D\varphi_{bi}\varphi_{bt}^{-1})\left(\chi_{ab}-v_b^j\frac{\partial\chi_{ab}}{\partial v_b^j}\right)-\right. \\
&\left.\left.\varphi_{bi}\varphi_{bt}^{-1}\left(\frac{\partial\chi_{ab}}{\partial t}+c\frac{\partial\chi_{ab}}{\partial\theta}\right)\right]\right\}, \qquad (3.15)
\end{aligned}$$

$$\begin{aligned}
J_i &= \epsilon_{ijk}\sum_a m_a\Gamma_a x_{aj}v_{ak} - \\
&\epsilon_{ijk}\sum_{a<}\sum_b\int d\theta\left\{\exp\left[\frac{\theta}{c}\left(D_b+\frac{\partial}{\partial t}\right)\right]x_{aj}\left[\frac{\partial\chi_{ab}}{\partial v_a^k}-\right.\right. \\
&\left.\varphi'_{ak}(\varphi'_{at})^{-1}\left(\chi_{ab}-v_a^l\frac{\partial\chi_{ab}}{\partial v_a^l}\right)\right]+ \\
&\exp\left(-\frac{\theta}{c}D_a\right)x_{bj}\left[\frac{\partial\chi_{ab}}{\partial v_b^k}-\varphi_{bk}\varphi_{bt}^{-1}\left(\chi_{ab}-v_b^l\frac{\partial\chi_{ab}}{\partial v_b^l}\right)\right]+ \\
&\frac{\theta}{c}\int_0^1 d\tau\exp\left(\frac{\theta}{c}(\tau D-D_a)\right)\left[x_{bj}\frac{\partial\chi_{ab}}{\partial x_b^k}+v_{bj}\frac{\partial\chi_{ab}}{\partial v_b^k}+\right. \\
&\left.\left.(D(x_{bk}\varphi_{bj}\varphi_{bt}^{-1}))\left(\chi_{ab}-v_b^l\frac{\partial\chi_{ab}}{\partial v_b^l}\right)+x_{bk}\varphi_{bj}\varphi_{bt}^{-1}\left(\frac{\partial\chi_{ab}}{\partial t}+c\frac{\partial\chi_{ab}}{\partial\theta}\right)\right]\right\}, \qquad (3.16)
\end{aligned}$$

$$K_i = \sum_a m_a\Gamma_a(-tv_{ai}+x_{ai}) +$$

$$\frac{1}{c}\sum_{a<}\sum_{b}\int d\theta\left\{\exp\left[\frac{\theta}{c}\left(D_b+\frac{\partial}{\partial t}\right)\right]\left[\varphi'_a\frac{\partial\chi_{ab}}{\partial v_a^i}+\right.\right.$$
$$\left.(x_{ai}-\varphi'_a\varphi'_{ai})(\varphi'_{at})^{-1}\left(\chi_{ab}-v_a^j\frac{\partial\chi_{ab}}{\partial v_a^j}\right)\right]+\exp\left(-\frac{\theta}{c}D_a\right)\left[\varphi_b\frac{\partial\chi_{ab}}{\partial v_b^i}+\right.$$
$$\left.(x_{bi}-\varphi_b\varphi_{bi})\varphi_{bt}^{-1}\left(\chi_{ab}-v_b^j\frac{\partial\chi_{ab}}{\partial v_b^j}\right)\right]+$$
$$\frac{\theta}{c}\int_0^1 d\tau\exp\left(\frac{\theta}{c}(\tau D-D_a)\right)\left[\varphi_b\frac{\partial\chi_{ab}}{\partial x_b^i}+(D\varphi_b)\frac{\partial\chi_{ab}}{\partial v_b^i}+\right.$$
$$(D(x_{bi}-\varphi_b\varphi_{bi})\varphi_{bt}^{-1})\left(\chi_{ab}-v_b^j\frac{\partial\chi_{ab}}{\partial v_b^j}\right)+$$
$$\left.\left.(x_{bi}-\varphi_b\varphi_{bi})\varphi_{bt}^{-1}\left(\frac{\partial\chi_{ab}}{\partial t}+c\frac{\partial\chi_{ab}}{\partial\theta}\right)\right]\right\}. \tag{3.17}$$

Substituting the definitions (1.37) or (1.40), (1.42) of instant or isotropic forms of dynamics into expressions (3.14), (3.17), we can easily find conserved quantities in these forms of relativistic dynamics. In [9] for the case of the instant form of dynamics these quantities were derived and written down in a slightly different manner with introduction of the formal inverse operator D^{-1}. Such expressions are better adapted to the investigation of expansions in the parameter c^{-1}. The result obtained here, namely formulae (3.14)–(3.17), is useful for the consideration of the coupling constant expansion. The equivalence of both results can be directly demonstrated by formal calculation of the τ-integrals in the expressions above.

Conclusion

The main purpose of this work was to examine the symmetry properties of the single-time nonlocal Lagrangian description that arises from the Fokker-type action integrals in various forms of relativistic dynamics. Evidently, the preservation of the Poincaré-invariance under transition from manifestly invariant action to its single-time form was easy to predict. The essential physical and technical complications in treating relativistic single-time Lagrangians consist in the fact that they must depend on the derivatives of arbitrary high order [13, 10]. A suitable tool for dealing with such objects is provided by the theory of jet spaces and Lie-Bäcklund transformations [22–24]. In this respect the basic result contained in this report is that nonlocal structure (1.27), (1.30) of the interaction Lagrangians provides us with a useful ansatz for solution of the Poincaré-invariance condition in any form of relativistic dynamics. As a consequence of this result we find explicit expressions for the ten conserved quantities: energy, momentum, angular momentum, and center-of-mass integral of motion.

From the physical point of view the obtained Lagrangians and conserved quantities may be considered as a starting point for performing various approximations (for example, the expansions in powers of coupling constant or c^{-1}). Such approximations allow us to return to more convenient predictive description in $6N$-dimensional phase space [10–12]. On the other hand, we can use a nonlocal Lagrangian structure in the investigation of invariance under other groups which are of interest in physics (Galilei, de Sitter or full conformal groups).

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