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Symmetry Properties of Generalized Gas Dynamics Equations

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Abstract

We describe a class of generalized gas dynamics equations invariant under the extended Galilei algebra $A\tilde{G}(1,n)$.

Symmetry properties of the gas dynamics equations

$$\left\{ \begin{array}{l} \vec{u}_0 + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}p = 0, \\ \rho_0 + \mathrm{div}\,(\rho\vec{u}) = 0, \\ p = f\,(\rho)\,, \end{array} \right. \tag{1}$$

where $\vec{u}$ is the velocity, ρ is the density, p is the pressure of gas, were investigated in [1]. As has been shown in [1], the system (1) has the widest symmetry when $f(\rho) = \lambda\rho^{\frac{n+2}{n}}$ ($\lambda = \mathrm{const}, n$ is the quantity of space variables $\vec{x} \in R_n$). In this case a basis of the maximum invariant algebra of the equation (1) is represented by the operators

$$\begin{array}{ll} \partial_0, \quad \partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, & \\ D_1 = x_0\partial_0 + x_a\partial_a, \qquad D_2 = x_0\partial_0 - n\rho\partial_\rho - u^a\partial_{u^a}, & \\ G_a = x_0\partial_a + \partial_{u^a}, \qquad \Pi = x_0\left(x_0\partial_0 + x_a\partial_a - n\rho\partial_p - u^a\partial_{u^a}\right), & \end{array} \tag{2}$$

where $a, b = \overline{1,n}$.

We shall call the algebra (2) the extended Galilean algebra and designate it by $A\tilde{G}(1,n)$. Other models of gas conduct are well-known except the system (1) (see, for example, [2]). Usually the first and second equations of the system (1) are invariables and the third equation has any form. For this reason we have the problem of finding the function

$$S = S\left(x_0, \vec{x}, \vec{u}, \rho, p, \vec{u}_0, \vec{u}_a, \rho_0, \rho_a, p_0, p_a\right), \tag{3}$$

where $a = \overline{1,n}, \vec{u}_a = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}\,, \rho_a = \vec{\nabla}\rho, p_a = \vec{\nabla}p, \vec{u}_a = \dfrac{\partial\vec{u}}{\partial x_a}$, with which the system

$$\left\{ \begin{array}{l} \vec{u}_0 + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}p = 0, \\ \rho_0 + \mathrm{div}\,(\rho\vec{u}) = 0, \\ S = 0 \end{array} \right. \tag{4}$$

is invariant with respect to the algebra $A\tilde{G}(1,n)$. It follows from the invariance with respect to the operators ∂_0, ∂_a that this system has the form

$$\left\{\begin{array}{l} \vec{u}_0 + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}p = 0, \\ \rho_0 + \mathrm{div}\,(\rho\vec{u}) = 0, \\ \rho_0 + F\left(\vec{u}, \rho, p, \vec{u}_a, \rho_a, p_a\right) = 0. \end{array}\right. \tag{5}$$

The infinitesimal operator of the algebra $A\tilde{G}(1,n)$ has the following form

$$X = d_\mu\partial_\mu + c_{ab}J_{ab} + g_aG_a + \kappa_1D_1 + \kappa_2D_2 + a\Pi + \eta\left(x_0, \vec{x}, \vec{u}, \rho, p\right)\partial_\rho \tag{6}$$

Using invariance of the first equation of the system (5) with respect to the operator (6), we obtain

$$\eta = -\left(n+2\right)\left(ax_0 + \kappa_2\right)p. \tag{7}$$

This means that all operators of the algebra (2) must have such a form except

$$D_2' = D_2 - (n+2)p\partial_p, \quad \Pi' = \Pi - (n+2)x_0p\partial_p. \tag{8}$$

Demanding that the third equation of the system (5) be invariant with respect to the Galilean operators, we obtain

$$\frac{\partial F}{\partial u^a} = p_a. \tag{9}$$

Hence

$$F = \left(\vec{u}\vec{\nabla}\right)p + \Phi\left(\rho, p, \vec{u}_a, \rho_a, p_a\right). \tag{10}$$

We assume that

$$\Phi = \Phi\left(\rho, p, \vec{\nabla}\vec{u}, \vec{\nabla}\rho, \vec{\nabla}p\right). \tag{11}$$

It follows from invariance with respect to the rotations J_{ab} that

$$\Phi = \Phi\left(\rho, p, w_1, w_2, w_3, w_4\right), \tag{12}$$

where

$$w_1 = \vec{\nabla}\vec{u}, w_2 = \left(\vec{\nabla}\rho\right)^2, w_3 = \left(\vec{\nabla}\rho\right)\left(\vec{\nabla}p\right), w_4 = \left(\vec{\nabla}p\right)^2. \tag{13}$$

Substituting (10)–(13) in the third equation of the system (5), we have

$$p_0 + (\vec{u}\cdot\vec{\nabla})p + \Phi\left(\rho, p, w_1, w_2, w_3, w_4\right) = 0. \tag{14}$$

Now let us consider the invariance of the equation (14) with respect to the operators D_1, D_2', Π'. For each of them we obtain the equation for the function Φ

$$\begin{array}{ll} D_1: & w_1\Phi_1 + 2w_2\Phi_2 + 2w_3\Phi_3 + 2w_4\Phi_4 - \Phi = 0, \\ D_2': & n\rho\Phi_\rho + (n+2)p\Phi_p + w_1\Phi_1 + 2nw_2\Phi_2 + 2(n+1)w_3\Phi_3 + \\ & \quad 2(n+2)w_4\Phi_4 - (n+3)\Phi = 0; \\ \Pi': & n\rho\Phi_\rho + (n+2)p\Phi_p + 2w_1\Phi_1 + 2(n+1)w_2\Phi_2 + 2(n+2)w_3\Phi_3 + \\ & \quad 2(n+3)w_4\Phi_4 - (n+4)\Phi + n\Phi_1 - (n+2)p = 0. \end{array} \tag{15}$$

The function

$$\Phi = \frac{n+2}{n} p \operatorname{div} \vec{u} - \mid \vec{\nabla} p \mid p^{\frac{1}{n+2}} H \left(\rho p^{-\frac{n}{n+2}}, \frac{\vec{\nabla}\rho\vec{\nabla}p}{\left(\vec{\nabla}\rho\right)^2} p^{-\frac{2}{n+2}}, \frac{\mid \vec{\nabla} p \mid}{\mid \vec{\nabla}\rho \mid} p^{-\frac{2}{n+2}} \right) \tag{16}$$

is a general solution of the system (15).

Thus we have proved the following

Theorem. *The system (5) is invariant with respect to the extended Galilean algebra $A\tilde{G}(1,n)$ (2), (8) when it has the form*

$$\begin{cases} \vec{u}_0 + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}p = 0, \\ \rho_0 + \operatorname{div}(\rho\vec{u}) = 0, \\ p_0 + (\vec{u}\cdot\vec{\nabla})p + \frac{n+2}{n} p \operatorname{div} \vec{u} = \\ \qquad \mid \vec{\nabla} p \mid p^{\frac{1}{n+2}} H \left(\rho p^{-\frac{n}{n+2}}, \frac{\vec{\nabla}\rho\vec{\nabla}p}{\left(\vec{\nabla}\rho\right)^2} p^{-\frac{2}{n+2}}, \frac{\mid \vec{\nabla} p \mid}{\mid \vec{\nabla}\rho \mid} p^{-\frac{2}{n+2}} \right) \end{cases} \tag{17}$$

where H is an arbitrary smooth function.

Notation 1. At $H = 0$ the result of the theorem is the same as one obtained by Ovsyannikov in [2].

Notation 2. By substitution

$$p = \lambda \frac{n}{n+2} P^{\frac{n+2}{n}}, \quad \lambda = \text{const} \tag{18}$$

the system (17) reduces to

$$\begin{cases} \vec{u}_0 + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{\lambda}{\rho} P^{\frac{2}{n}} \vec{\nabla} P = 0, \\ \rho_0 + \operatorname{div}(\rho\vec{u}) = 0, \\ P_0 + \operatorname{div}(P\vec{u}) = P^{\frac{1}{n}} \mid \vec{\nabla} P \mid f \left(\frac{P}{\rho}, \frac{\vec{\nabla}\rho\vec{\nabla}P}{\left(\vec{\nabla}\rho\right)^2}, \frac{\mid \vec{\nabla} P \mid}{\mid \vec{\nabla}\rho \mid} \right). \end{cases} \tag{19}$$

References

[1] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993, 436p.

[2] Ovsyannikov L.W., Lectures on the Basis of Gas Dynamics, Moscow, Nauka, 1981, 368p.