Nonlinear Mathematical Physics 1996, V.3, N 3–4, 435–440.

Symmetry Analysis of the Multidimensional Polywave Equation

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Abstract

We present symmetry classification of the polywave equation $\Box^l u = F(u)$. It is established that the equation in question is invariant under the conformal group $C(1,n)$ iff $F(u) = \lambda e^u$, $n+1 = 2l$ or $F(u) = \lambda u^{(n+1+2l)/(n+1-2l)}$, $n+1 \neq 2l$. Symmetry reduction for the biwave equation $\Box^2 u = \lambda u^k$ is carried out. Some exact solutions are obtained.

The nonlinear wave equation

$$\Box u = F(u) \tag{1}$$

is known to describe a scalar spinless uncharged particle in the quantum field theory. Symmetry properties of Eq.(1) were studied in [1, 2, 3] and wide classes of its exact solutions with certain specific values of the function $F(u)$ were obtained in [1, 2, 4, 5, 6].

It is suggested in [7] to describe different physical processes with the help of nonlinear partial equations of high order. So we consider the generalized wave equation, namely, the polywave equation

$$\Box^l u = F(u). \tag{2}$$

Here $\Box^l = \Box(\Box^{l-1})$, $l \in \mathrm{N}$; $\quad \Box = \dfrac{\partial^2}{\partial x_0^2} - \dfrac{\partial^2}{\partial x_1^2} - \ldots - \dfrac{\partial^2}{\partial x_n^2}$ is a d'Alembertian in the pseudo-Euclidean space $R(1,n)$ with the metric tensor $g_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$, $\mu, \nu = \overline{0, n}$; $F(u)$ is an arbitrary smooth function and $u = u(x)$ is a real function.

Symmetry of Eq.(2) when l is not positive integer is described in [1, 8].

Group properties of Eq.(2) are investigated in [1, 9, 10], where the conformal invariance of the equation is ascertained.

We establish that the conformal group is the maximal invariance group of the equation in question [11]. It occurs that the group properties of differential equation (2) are virtually the same as those of standard wave equation (1).

Let us note that as the equation (2) is of high order and we consider it in a multidimensional space (l, n are arbitrary positive integers), one cannot use existing symbolic manipulation programs for studying its symmetry [12]. The maximal symmetry group of Eq.(2) is constructed by means of the infinitesimal algorithm of S. Lie [7, 13].

Results of symmetry classification of Eq.(2) are given in the following statements. The case $l = 1,\ n = 1$ has been studied earlier (see [2]), that is why we consider the case $l + n > 2$.

Lemma 1 *The maximal invariance group of Eq.(2) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators*

$$P_\mu = \frac{\partial}{\partial x_\mu}, \qquad J_{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}, \qquad \mu,\nu = \overline{0,n}. \tag{3}$$

Here and further summation over repeated indices from 0 to n is understood; $x_\mu = x^\nu g_{\mu\nu},\ \mu,\nu = \overline{0,n}$.

Theorem 1 *All the equations of type (2) admitting a more extended invariance algebra than the Poincaré algebra $AP(1,n)$ are equivalent to one of the following:*

$$1.\quad \Box^l u = \lambda_1 u^k, \qquad \lambda_1 \neq 0,\ k \neq 0,1; \tag{4}$$
$$2.\quad \Box^l u = \lambda_2 e^u, \qquad \lambda_2 \neq 0; \tag{5}$$
$$3.\quad \Box^l u = \lambda_3 u, \qquad \lambda_3 \neq 0; \tag{6}$$
$$4.\quad \Box^l u = 0. \tag{7}$$

Here $\lambda_1,\ \lambda_2,\ \lambda_3$ are arbitrary constants.

Theorem 2 *The symmetry of the Eqs.(4)–(7) is described in the following way:*

1. (a) *The maximal invariance group of Eq.(4) when $k \neq (n+1+2l)/(n+1-2l)$, $k \neq 0,1$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (3) and*

   $$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{2l}{1-k} u \frac{\partial}{\partial u}.$$

   (b) *The maximal invariance group of Eq.(4) when $k = (n+1+2l)/(n+1-2l)$, $n+1 \neq 2l$ is the conformal group $C(1,n)$ generated by the operators (3) and*

   $$\begin{aligned} D^{(1)} &= x_\mu \frac{\partial}{\partial x_\mu} + \frac{2l-n-1}{2} u \frac{\partial}{\partial u}, \\ K^{(1)}_\mu &= 2x^\mu D^{(1)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \end{aligned} \tag{8}$$

2. (a) *The maximal invariance group of Eq.(5) when $n \neq 2l-1$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (3) and*

   $$D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} - 2l \frac{\partial}{\partial u}.$$

   (b) *The maximal invariance group of Eq.(5) when $n = 2l-1$ is the conformal group $C(1,n)$ generated by the operators (3) and*

   $$K^{(2)}_\mu = 2x^\mu D^{(2)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}.$$

3. *The maximal invariance group of Eq.(6) is generated by the operators (3) and*

$$Q = h(x)\frac{\partial}{\partial u}, \qquad I = u\frac{\partial}{\partial u},$$

where $h(x)$ is an arbitrary solution of Eq.(6).

4. *The maximal invariance group of Eq.(7) is generated by the operators (3), (8) and*

$$Q = q(x)\frac{\partial}{\partial u}, \qquad I = u\frac{\partial}{\partial u},$$

where $q(x)$ is an arbitrary solution of Eq.(7).

Since proofs of Lemma 1 and Theorems 1,2 require very cumbersome computations, we omit them.

An important consequence of the above theorems is the following statement.

Corollary *Provided $n+1=2l$, there exist two inequivalent representations of a Lie algebra of the conformal group on the solution set of equation (7) [7, 1, 8]:*

$$\begin{array}{llrcl} 1. & P^{(1)}_\mu = P_\mu = \partial_{x_\mu}, & J^{(1)}_{\mu\nu} & = & J_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \\ & D^{(1)} = x_\mu \partial_{x_\mu}, & K^{(1)}_\mu & = & 2x^\mu D^{(1)} - (x_\nu x^\nu)\partial_{x_\mu}; \end{array}$$

$$\begin{array}{llrcl} 2. & P^{(2)}_\mu = P_\mu = \partial_{x_\mu}, & J^{(2)}_{\mu\nu} & = & J_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \\ & D^{(2)} = x_\mu \partial_{x_\mu} + \partial_u, & K^{(2)}_\mu & = & 2x^\mu D^{(2)} - (x_\nu x^\nu)\partial_{x_\mu}. \end{array}$$

As follows from the foregoing statements, a set of equations of type (2) which are invariant under the extended Poincaré group $\widetilde{P}(1,n)$ is exhausted by equations (4), (5), (7). Besides, the nonlinear equation (2) is invariant under the conformal group $C(1,n)$ iff it is equivalent to the following

$$\begin{array}{lrcll} 1. & \Box^l u & = & \lambda_1 u^{\frac{n+1+2l}{n+1-2l}}, & n+1 \neq 2l; \\ 2. & \Box^l u & = & \lambda_2 e^u, & n+1 = 2l. \end{array}$$

From our point of view it is of great interest to make use of symmetry properties of PDE (4)–(7) in order to construct some exact solutions by analogy with what have been made in [1, 7, 14] for the nonlinear wave equation. Here we consider the biwave equation in the two-dimensional space $R(1,1)$:

$$\Box^2 u = \lambda u^k, \qquad \Box = \partial^2_{x_0} - \partial^2_{x_1} \tag{9}$$

which is invariant under the extended Poincaré group $\widetilde{P}(1,1)$.

Making use of inequivalent one–dimensional subalgebras of the conformal algebra $AC(1,1)$ [6], one can obtain the following $C(1,1)$-inequivalent ansatzes which reduce equation (9) to ordinary differential equations. For each case the reduced equations are given as

1. $\quad u = (x_0 - x_1)^{\frac{2}{1-k}} \varphi(\omega), \qquad \omega = x_0 + x_1;$

$$\frac{1+k}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{32}\varphi^k;$$

2. $u = (x_0+x_1)^{\frac{4}{(1-k)(\alpha+1)}}\varphi(\omega), \qquad \omega = (x_0-x_1)(x_0+x_1)^{\frac{\alpha-1}{\alpha+1}}, \quad \alpha \neq -1;$

$$(\alpha-1)^2\varphi^{(4)}\omega^2 + 2(\alpha-1)(\alpha+1)^2\left(\frac{3k+1}{1-k}+2\alpha\right)\omega\varphi^{(3)} +$$

$$2\left(\alpha^2-4\alpha+3+\frac{6\alpha-10}{1-k}+\frac{8}{(1-k)^2}\right)\varphi^{(2)} = \frac{\lambda}{16}(\alpha+1)^2\varphi^k;$$

3. $u = \exp\Big(\frac{4}{k-1}(x_1-x_0)\Big)\varphi(\omega), \qquad \omega = (x_0+x_1+\frac{1}{2})\exp\Big(2(x_1-x_0)\Big);$

$$\varphi^{(4)}\omega^2 + \frac{5k-1}{k-1}\varphi^{(3)}\omega + \frac{4k^2}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{64}\varphi^k;$$

4. $u = \varphi(\omega), \qquad \omega = x_0^2 - x_1^2;$

$$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^k;$$

5. $u = \varphi(\omega), \qquad \omega = x_1;$

$$\varphi^{(4)} = \lambda\varphi^k;$$

6. $u = \varphi(\omega), \qquad \omega = x_0 + x_1;$

$$\lambda\varphi^k = 0.$$

Finding some solutions of the reduced equations leads us to the following solutions of Eq.(9):

$$u = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\Big((x_0+x_1+c_1)(x_0-x_1+c_2)\Big)^{-\frac{2}{k-1}}, \qquad k \neq -1,$$

$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}(x_0+c_3)^{\frac{4}{1-k}}, \qquad k \neq -1, -3, -\frac{1}{3},$$

$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}(x_1+c_4)^{\frac{4}{1-k}}, \qquad k \neq -1, -3, -\frac{1}{3},$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Note that Eq.(1) has analogous solutions (see [1]).

It follows from Theorem 2 that Eq.(9) when $k=-3$ is invariant under the conformal group $C(1,1)$. So using $C(1,1)$–inequivalent ansatzes, one can carry out the symmetry reduction of the equation

$$\Box^2 u = \lambda u^{-3}. \tag{10}$$

The corresponding results are presented below:

1. $u = \varphi(\omega), \qquad \omega = x_0$ or $\omega = x_1;$

$$\varphi^{(4)} = \lambda\varphi^{-3};$$

2. $u = \varphi(\omega), \qquad \omega = x_0^2 - x_1^2;$

$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$

3. $u = (x_0 + x_1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 - x_1;$

$\varphi^{(2)} = -\frac{\lambda}{4}\varphi^{-3};$

4. $u = (x_0 + x_1)^{\frac{1}{\alpha+1}}\varphi(\omega), \qquad \omega = (x_0 - x_1)(x_0 + x_1)^{\frac{\alpha-1}{\alpha+1}};$

$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \frac{(\alpha-2)(2\alpha-1)}{(\alpha-1)^2}\varphi^{(2)} = \frac{\lambda}{16}\frac{(\alpha+1)^2}{(\alpha-1)^2}\varphi^{-3}, \ \alpha > 1;$

5. $u = \exp(x_0 - x_1)\varphi(\omega), \qquad \omega = (x_0 + x_1 + \frac{1}{2})\exp(-2(x_0 - x_1));$

$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \frac{9}{4}\varphi^{(2)} = \frac{\lambda}{64}\varphi^{-3};$

6. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 + x_1;$

$\varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$

7. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 + x_1 + \arctan(x_0 - x_1);$

$\varphi^{(4)} + \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$

8. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 + x_1 + \frac{1}{2}\ln\frac{1 + x_0 - x_1}{1 - x_0 + x_1};$

$\varphi^{(4)} - \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$

9. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}(x_0 + x_1)^{\frac{1}{2}}\varphi(\omega),$

$\omega = \ln(x_0 + x_1) - \beta\arctan(x_1 - x_0);$

$4\beta^2\varphi^{(4)} + (4 - \beta^2)\varphi^{(2)} - \varphi = \frac{\lambda}{4}\varphi^{-3}, \ \beta > 0;$

10. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}((x_0 + x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega),$

$\omega = (\gamma - 1)\arctan(x_0 - x_1) + (\gamma + 1)\arctan(x_0 + x_1);$

$(\gamma^2 - 1)^2\varphi^{(4)} + 2(\gamma^2 + 1)\varphi^{(2)} + \varphi = \frac{\lambda}{16}\varphi^{-3}, \ 0 \le \gamma < 1.$

Integrating the reduced equations, we can find a number of exact solutions of the nonlinear biwave equation (10). Here we present some exact solutions of this equation with use of the ansatzes 3 and 6:

$$\begin{aligned} &1. \quad u = \pm\lambda^{\frac{1}{4}}\left(x_0^2 - x_1^2\right)^{1/2}, \\ &2. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{c_1}\right)^{\frac{1}{4}}|(x_0 - x_1)^2 - c_1|^{1/2}\left(x_0 + x_1\right)^{1/2}, \\ &3. \quad u = \pm\frac{1}{2}\left(\frac{\lambda}{c_2}\right)^{\frac{1}{4}}\left((x_0 - x_1)^2 + 1\right)^{1/2}|(x_0 + x_1)^2 + c_2|^{1/2}, \end{aligned} \tag{11}$$

where $c_1, \ c_2$ are arbitrary constants.

In conclusion let us note that we can obtain solutions (11) of Eq.(10) using the following ansatz

$$u = \varphi_1(\omega_1)\varphi_2(\omega_2), \qquad \omega_1 = x_0 + x_1, \ \ \omega_2 = x_0 - x_1,$$

that reduces equation (10) to the system of ordinary differential equations for unknown functions $\varphi_1(\omega_1)$ and $\varphi_2(\omega_2)$, namely,

$$\varphi_1^{(2)} = \frac{c}{4}\varphi_1^{-3}, \qquad \varphi_2^{(2)} = \frac{\lambda}{4c}\varphi_2^{-3}; \tag{12}$$

where c is an arbitrary constant.

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