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Conditional Symmetry and Exact Solutions of a Nonlinear Galilei-Invariant Spinor Equation

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Abstract

Reduction of a nonlinear system of differential equations for spinor field is studied. The ansatzes obtained are shown to correspond to operators of conditional symmetry of these equations.

Let us consider the nonlinear system of differential equations (DE) for a spinor field

$$\Big\{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4) + f_1(\bar\psi\psi, \bar\psi(\gamma_0+\gamma_4)\psi) + f_2(\bar\psi\psi, \psi(\gamma_0+\gamma_4)\psi)(\gamma_0+\gamma_4)\Big\}\psi = 0, \tag{1}$$

where $\psi=\psi(t,\bar x)$ is four-component complex function, $\gamma_0,\ldots,\gamma_4$ — Dirac matrix (4×4), $\partial_t = \dfrac{\partial}{\partial t}, \partial_a = \dfrac{\partial}{\partial x_a}, a=\overline{1,3}; f_1 f_2 \subset C_2(R^2, C^1); m = \text{const}$. It is known that this system is invariant under the Galilei group. For finding exact solutions of system (1) the ansatz

$$\psi(t,\bar x) = \exp i\theta_0 + \theta_a\gamma_a(\gamma_0+\gamma_4)\varphi(\omega), \tag{2}$$

is used [1]. Here $\theta_0,\ldots,\theta_3,\omega$ are real scalar functions, which are chosen so that the substitution (2) into system (1) would lead to a system of ordinary DE for the function $\varphi(\omega)$ [2]. This substitution leads to the following system of nonlinear DE for the functions $\theta_\mu(t,\bar x),\ \omega(t,\bar x)$:

$$\begin{array}{ll} \text{rot}\,\bar\theta = \bar F(\omega), & \text{div}\,\bar\theta = F_4(\omega), \\ \partial_t\theta_0 + 2\theta_a\partial_a\theta_a + 4m\theta_a\theta_a = F_5(\omega), & \partial_a\theta_0 + 4m\theta_a = F_{5+a}(\omega), \\ \partial_t\omega + 2\theta_a\partial_a\theta_0 = F_9(\omega), & \partial_a\omega = F_{9+a}(\omega). \end{array} \tag{3}$$

Here and further, summation is meant over recurring indices, F_1–F_{12} are arbitrary smooth real functions, $a=1,2,3$.

We find the solutions which are determined up to equivalence under the Galilei group. Due to arbitrariness of $\varphi(\omega)$, the substitution θ_μ,ω and $\theta_\mu + h_\mu(\omega), h(\omega)$ into (2) gives the same ansatz for $\psi(t,\bar x)$. That is why we consider such solutions as equivalent.

Theorem. *The general solution of the system of ordinary DE (3), which is determined up to equivalence introduced above, is defined by one of the following formulae.*

1. $m=0$

$$\begin{array}{ll} 1) & \omega = x_1 + W_1(t), \\ & \theta_0 = C_3(x_2 - 2W_2(t)) + C_4(x_3 - 2W_3(t)) + C_5 t, \\ & \theta_1 = -\frac{1}{2}\dot W_1(t), \\ & \theta_2 = -\alpha(C_3x_2 + C_4x_3) + \dot W_2(t) + C_1x_2, \\ & \theta_3 = \alpha(C_3x_3 + C_4x_2) + \dot W_3(t) + C_2x_2, \\ & \alpha = (C_1C_3 + C_2C_4)(C_3^2 + C_4^2)^{-1}, \end{array} \tag{4}$$

2) $\omega = x_1 + W_0(t)$,
$\theta_0 = C_3 t$,
$\theta_1 = -\frac{1}{2}\dot{W}_0(t)$,
$\theta_2 = U(z,t) + U(z^*,t) + C_1 x_2$,
$\theta_3 = i(U(z,t) - U(z^*,t)) + C_2 x_2$,
$z = x_2 + i x_3$,

3) $\omega = t$,
$\theta_0 = x_b g_b(t), \quad b = \overline{1,3}$,
$\theta_a = \varepsilon_{abc} h_b(t) x_c + \partial_a \Phi + W(t) x_a$,

here the function $\Phi = \Phi(t, \bar{x})$ is defined by the following expressions
a) for $g_1 = g_2 = g_3$, $\partial_a \partial_a \Phi = 0$,
b) for $g_2 = 0, \quad g_3 \neq 0$

$$\Phi = g_3^{-1}\left[r_1 x_1 x_3 + r_2 x_2 x_3 + r_4 x_3 + \frac{1}{2} r_3 x_3^2 - \frac{1}{2} g_3^{-1} g_1 r_1 x_3^2 + \frac{1}{2}(g_3^{-1} g_1 r_1 - r_3) x_2^2\right] + U(z,t) + U(z^*,t),$$

$$z = (g_1^2 + g_3^2)^{-\frac{1}{2}}(g_1 x_3 - g_3 x_1) + i x_2,$$

c) for $g_1 \neq 0, \quad g_3 = 0$

$$\Phi = g_1^{-1}\left[\frac{1}{2} r_1 x_1^2 - \frac{1}{2} g_1^{-1} g_2 r_2 x_1^2 + r_2 x_1 x_2 + r_3 x_1 x_2 + r_4 x_1 + \frac{1}{2}(g_1^{-1} g_2 r_2 - r_1) x_3^2\right] + U(z,t) + U(z^*,t),$$

$$z = (g_1^2 + g_3^2)^{-\frac{1}{2}}(g_1 x_3 - g_3 x_1) + i x_2,$$

d) for $g_1^2 + g_2^2 \neq 0, g_3 = 0$

$$\Phi = \frac{1}{2} g_3^{-2} r_1 (2 g_3 x_1 x_3 - g_1 x_3^2) g_3 r_2 (2 g_3 x_2 x_3 - g_2 x_3^2) + \frac{1}{2} g_3^{-1}(r_3 x_3^2 + 2 r_4 x_3) + \frac{1}{2} g_3^2 (g_1^2 + g_2^2)^{-1}(r_1 g_1 + r_2 g_2 - r_3 g_3)(g_2 x_1 - g_1 x_2)^2 + U(z,t) + U(z^*,t),$$

$$z = [(g_1^2 + g_2^2)^{-1}(g_2^2 + g_3^2) - g_1^2 g_3^2 (g_1^2 + g_2^2)^{-2}]^{\frac{1}{2}}(g_2 x_1 - g_1 x_2) + i[g_1 g_3 (g_1^2 + g_2^2)^{-1}(g_2 x_1 - g_1 x_2) + g_3 x_2 - g_2 x_3];$$

In the formula a)–d)

$$r_1 = -\left(g_1 W + g_2 h_3 - h_2 g_3 + \frac{1}{2}\dot{g}_1\right), \quad r_2 = -\left(g_2 W + g_3 h_1 - g_1 h_3 + \frac{1}{2}\dot{g}_2\right),$$

$$r_3 = -\left(g_3 W + g_1 h_2 - g_2 h_1 + \frac{1}{2}\dot{g}_3\right), \quad r_4 = W_0(t).$$

2. $m \neq 0$

1) $\omega = x_1 + 4m)^{-1} C_5 t_1^2 + C_7 t$,
$\theta_0 = (2mC_7 + C_5 t)\omega + (C_3 - 4mC_1) x_2 + (C_4 - 4mC_2) x_3 - (12m)^{-1} C_5^2 t^3 - \frac{1}{2} C_5 C_7 t^2 + C_6 t$,
$\theta_1 = -(4m)^{-1} C_5 t - \frac{1}{2} C_7$,
$\theta_2 = C_1$,
$\theta_3 = C_2$;

2) $\omega = t,$
$\theta_0 = -2mW_0 x_a x_a + R_a x_a - 4m(T_{ab} x_a x_b + T_a x_a),$
$\theta_a = W_0 x_a + 2T_{ab} x_b + T_a,$

where $R_a(t), T_{ab}(t), T_a(t)$ are real functions which satisfy the system of ordinary DE

$$\begin{aligned}
&\dot{T}_0 + 2\dot{T}_{aa} + 2(T_{1a}^2 + T_{2a}^2 + T_{3a}^2) + 8W_0 T_{aa} = 0.\\
&\dot{T}_{ab} + 4(T_{1a}T_{1b} + T_{2a}T_{2b} + T_{3a}T_{3b}) + 4W_0 T_{ab} = 0, \qquad a \neq b,\\
&\dot{R}_a - 4m\dot{T}_a - 8mW_0 - 16mT_{ab}T_b + 4T_{ab} + 2W_0 R_a = 0
\end{aligned}$$

(no summation over a), besides $T_{ab} = T_{ba}, T_{11} + T_{22} + T_{33} = 0$.

In the formula (4) $g_a(t), h_a(t)$ are arbitrary smooth functions. U is an arbitrary analytical with respect to z function, $C_1, C_2, \ldots, C_7$ are constants.

The substitution of formula (4) into expression (2) gives a collection of ansatzes for the field $\psi(t, \bar{x})$ which reduce system (1) to systems of ordinary DE

$$\begin{aligned}
1.\quad &1)\quad i\gamma_1\dot{\varphi} + i[(C_2\gamma_1 - C_1 - iC_5)(\gamma_0+\gamma_4) - iC_3\gamma_2 + iC_4\gamma_3]\varphi = R;\\
&2)\quad i\gamma_1\dot{\varphi} + i(C_2\gamma_1 - C_1 - iC_5)(\gamma_0+\gamma_4)\varphi = R;\\
&3)\quad -i(\gamma_0+\gamma_4)\dot{\varphi} + i[(2h_a + ig_a)\gamma_a - (3W + iW_0)(\gamma_0 + g_4)]\varphi = R;\\
2.\quad &1)\quad i\gamma_1\dot{\varphi} + [(C_5\omega + C_6 + m(C_7^2 - 4C_1^2 - 4C_2^2) + 2C_1C_3 + 2C_2C_4)\times\\
&\qquad (\gamma_0+\gamma_4) - C_3\gamma_2 - C_4\gamma_3 + m(\gamma_0 - \gamma_4)]\varphi = R;\\
&2)\quad -i(\gamma_0+\gamma_4)\dot{\varphi} + [-3iW_0(\gamma_0+\gamma_4) + (2R_aT_a - 4mT_aT_a)\times\\
&\qquad (\gamma_0+\gamma_4) + R_a\gamma_a + m(\gamma_0-\gamma_4)]\varphi = R,\\
&\qquad R = [f_1(\bar{\varphi}\varphi, \bar{\varphi}(\gamma_0+\gamma_4)\varphi) + f_2(\bar{\varphi}\varphi, \bar{\varphi}(\gamma_0+\gamma_4)\varphi)(\gamma_0+\gamma_4)]\varphi.
\end{aligned} \tag{5}$$

In general, the systems (5) cannot be integrated in quadratures. However, in some cases systems 1.2), 1.3) can be linearized and, consequently, their general solutions can be constructed. In particular, if

$$f_1 = iH_1,\ f_2 = H_2 + iH_3 \qquad \text{in the case 1.2)}$$

$$F_2 = 0,\ f_2 = H_2, h_a = g_a = W = W_0 \equiv 0 \qquad \text{in the case 1.3)},$$

where $H_i = H_i(\bar{\varphi}(\gamma_0+\gamma_4)\varphi)$ are smooth enough real functions, then

$$\bar{\varphi}(\gamma_0+\gamma_4)\varphi = \chi(\gamma_0+\gamma_4)\chi, \tag{6}$$

where χ is a constant four-component spinor.

The substitution of (6) into 1.2), 1.3) gives linear systems, whose solutions have the form

$$\begin{aligned}
\varphi = \exp\{&[\gamma_1(C_1\gamma_1 - C_2 - iC_5)(\gamma_0+\gamma_4) - \gamma_1 H_1(\chi(\gamma_0+\gamma_4)\chi) +\\
&\gamma_1(\gamma_0+\gamma_4)(ih_2(\bar{\chi}(\gamma_0+\gamma_4)\chi) - (\bar{\chi}(\gamma_0+\gamma_4)\chi))](x_1 + W(t))\}\,\chi,
\end{aligned} \tag{7}$$

$$\varphi = \exp\{iH - 2(\chi(\gamma_0+\gamma_4)\chi)^t\}\chi. \tag{8}$$

Substitution of (7) and 1.2) from (4), (8) and 1.3(a) from (4) in (2) gives the following classes of solutions for the equation (1)

$$\begin{aligned}\psi(t,\bar{x}) = \exp\Big\{ & iC_3t - \frac{1}{2}W(t)\gamma_1(\gamma_0+\gamma_4) + (U(z,t)+U(z^*,t)+C_2x_2)\times \\ & \gamma_2(\gamma_0+\gamma_4) + (i(U(z,t)+C_1x_1)\gamma_3(\gamma_0+\gamma_4)\Big\}\times \\ & \exp\Big\{[(C_1\gamma_1 - C_2 - iC_5)\gamma_1(\gamma_0+\gamma_4) - H_1(\bar{\chi}(\gamma_0+\gamma_4)\chi)\gamma_1 + \\ & (iH_2(\bar{\chi}(\gamma_0+\gamma_4)\chi) - H_3(\bar{\chi}(\gamma_0+\gamma_4)\chi))\gamma_1(\gamma_0+\gamma_4)](x_1+W(t))\Big\}\chi,\end{aligned} \tag{9}$$

$$\psi(t,\bar{x}) = \exp\Big\{ih_2(\bar{\chi}(\gamma_0+\gamma_4)\chi)t + \partial_a\varphi\gamma_a(\gamma_0+\gamma_4)\Big\}\chi, \tag{10}$$

where $z = x_2 + ix_3$, $W(t) \in C^2(R^1), U$ is an arbitrary analytic function with respect to z, Φ satisfies the three-dimensional Laplace equation $\partial_a\partial_a\Phi = 0$, χ is a constant four-component spinor.

It is necessary to emphasize that the ansatzes (2), where θ are defined by expressions 1.2), 1.3) from (4) and consequently solutions (9), (10) cannot be obtained by the traditional Lie approach. These ansatzes can be found by using a conditional symmetry of the non-linear equation (1). For these ansatzes let us write our differential operators for which $Q_a\psi = 0$, $a = \overline{1,3}$ are satisfied.

1. 2)
$$\begin{aligned} Q_1 &= \partial_1 - \dot{W}\partial_1 - iC_5 + \frac{1}{2}\ddot{W}\gamma_1(\gamma_0+\gamma_4) - \partial_1(U+U^*)\times \\ & \gamma_2(\gamma_0+\gamma_4) - i\partial_1(U-U^*)\gamma_3(\gamma_0+\gamma_4), \\ Q_2 &= \partial_2 - (\partial_zU + \partial_{z^*}U^* + C_2)\gamma_2(\gamma_0+\gamma_4) - (i\partial_zU - i\partial_{z^*}U^* + C_1)\gamma_3(\gamma_0+\gamma_4), \\ Q_3 &= \partial_3 - i(\partial_zU - \partial_{z^*}U^*)\gamma_2(\gamma_0+\gamma_4) + (\partial_z, U + \partial_{z^*}U^*)\gamma_3(\gamma_0+\gamma_4);\end{aligned}$$

1. 3)
$$\begin{aligned} Q_1 &= \partial_1 - [(\partial_1\partial_1\Phi + W)\gamma_1 + (\partial_1\partial_2\Phi + h_3)\gamma_2 + (\partial_1\partial_3\Phi - h_2)\gamma_3](\gamma_0+\gamma_4), \\ Q_2 &= \partial_2 - [(\partial_1\partial_2\Phi - h_3)\gamma_1 + (\partial_2\partial_2\Phi + W)\gamma_2 + (\partial_2\partial_3\Phi + h_1)\gamma_3](\gamma_0+\gamma_4), \\ Q_3 &= \partial_3 - [(\partial_1\partial_3\Phi - h_2)\gamma_1 + (\partial_2\partial_3\Phi - h_1)\gamma_2 + (\partial_3\partial_3\Phi + W)\gamma_3](\gamma_0+\gamma_4).\end{aligned}$$

The direct check-up shows that equation (1) is conditionally-invariant under operators Q_1, Q_2, Q_3.

References

[1] Fushchych W.I., Zhdanov R.Z., Symmetry and Exact Solutions of Nonlinear Spinor Equations, *Phys. Repts.,* 1989, V.46, N 2, 325–365.

[2] Zhdanov R.Z., Andreytsev A.Y., On non-Lie reduction of nonlinear Galilei-invariant equations for spinor field, *Dokl. AN Ukr.SSR. Ser. A,* 1990, 8–11.

[3] Andreytsev A.Y., On new exact solutions of nonlinear Galilei-invariant spinor equation for a zero mass particle, in: Symmetry Analysis of Equations of Mathematical Physics Kiev, Institute of Mathematics, 1992, 44–46.

[4] Fushchych W.I., Shtelen W.M., Serov M.I., Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics, Kiev, Naukova Dumka, 1989, 336p.

[5] Fushchych W.I., Zhdanov R.Z., Nonlinear Spinor Equations: Symmetry and Exact Solutions, Kiev, Naukova Dumka, 1992, 288p.